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NUMERICAL INVESTIGATION OF SOME NONLINEAR KDV EQUATIONS USING EXTENDED JACOBIAN ELLIPTIC FUNCTION EXPANSION METHOD

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ABSTRACT. In this paper, we investigate the Jacobian elliptic functions and give a clear definitions and properties of these functions. Some exact solutions for modified KdV equations are given by using the extended Jacobian elliptic function expansion method. Then, we compute some numerical solutions using semi Finite and finite difference methods. Stability of the numerical solutions are proved which prove the efficiently of these numerical solutions for such nonlinear partial differential equations.

1. INTRODUCTION

Nonlinear partial differential equations (NLPDEs) appear and have an essential role in addressing the problems in the physics of non-linear waves. We find NLPDEs in fluid dynamics [6, 14, 26], electromagnetic physics [15, 30], nonlinear optics [3, 9, 38], fluid mechanics [10, 12] etc.

In addition, a lot of methods are used to find exact solutions for NLPDEs. As examples, we can cite exp-function method [4, 11], the Riccati-Bernoulli sub-ODE method [2, 37], F-expansion method [28, 40], sine-cosine method [34, 35], homogeneous balance method [8, 31], Jacobi elliptic functions method [5, 19], $\tanh$- method [21, 33], extended $\tanh$-method [7, 36] and G/G'-expansion method [32, 39].

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We consider, in this paper, the mKdV equations

$$u_t + \alpha u^2 u_x - \beta u_{xxx} = 0. \tag{1.1}$$

The above equations arise in many domains as hyperbolic surfaces, ion acoustic solitons, slagmetallic bath interfaces, meandering ocean jets, dynamics of thin elastic rods, phonon in enharmonic lattices, traffic congestion, Schottky barrier transmission lines and Alfvén waves in collisionless plasmas [1, 13, 17, 20, 22, 24, 25, 27, 29, 41].

We use the extended Jacobian elliptic function expansion method (EJEFEM) to find exact solutions for the equations (1.1). We define these functions and explain the methodology of the method.

Numerical solutions are investigated to the equations (1.1), we use semi finite difference and finite differences methods [16, 23]. By studying the stability and analyzing the errors, we prove that the finite difference methods are practical and give a numerical solutions that converges to the exact ones.

The outlines of this paper are as follows. In section 2 we recall the Jacobian elliptic functions and describe the EJEFEM methodology. Section 3 is avoided to apply the EJEFEM method and find some exact solutions to the equation (1.1). In part 4, we apply the semi finite difference method to give numerical solution for the problem with Neumann boundary conditions. Section 5 is left to study the stability and section 6 for the errors. Finally, we give some conclusions and perspectives.

2. The Extended Jacobian Elliptic Function Expansion Method

In this section, we describe the extended Jacobian elliptic function expansion method (EJEFEM). For more details and other properties of the elliptic Jacobian functions, we can refer to [5, 18, 19]. The EJEFEM method can be used to find exact solutions for partial differential equations of the form

$$\mathcal{H}(u(x,t), u_t(x,t), u_x(x,t), u_{tt}(x,t), u_{xx}(x,t), u_{xt}(x,t), \ldots) = 0, \tag{2.1}$$

where $\mathcal{H}$ is a partial differential equation where the unknown function is $u(t,x)$ with two variables x and t.

First, we apply the transformation

$$\eta = k(x + wt) \text{ and } u = v(\eta) \tag{2.2}$$

to write the equation (2.1) as

$$\mathcal{G}(v, v', v'', v^{(3)}, \cdots) = 0. \tag{2.3}$$

Next, we look for an exact solution of equation (2.3) on the form

$$v(\eta) = a_0 + \sum_{j=1}^{N} f_i^{j-1}(\eta)\Big[a_j f_i(\eta) + b_j g_i(\eta)\Big],\ i = 1, 2, 3 \cdots \tag{2.4}$$

with a_j, b_j are in $\mathbb{R}$ and

$$\begin{aligned}
&f_1(\eta) = sn\eta, \quad g_1(\eta) = cn\eta, \quad f_2(\eta) = sn\eta, \quad g_2(\eta) = dn\eta,\\
&f_3(\eta) = ns\eta, \quad g_3(\eta) = cs\eta, \quad f_4(\eta) = ns\eta, \quad g_4(\eta) = ds\eta,\\
&f_5(\eta) = sc\eta, \quad g_5(\eta) = nc\eta, \quad f_6(\eta) = sd\eta, \quad g_6(\eta) = nd\eta,
\end{aligned} \tag{2.5}$$

where $sn\eta = sn(\eta, m)$ and $cn\eta = cn(\eta, m)$, $1 < m < 1$, are the Jacobian elliptic sine and cosine functions and $dn\eta = dn(\eta, m)$ is the third kind jacobian elliptic function. The other functions are defined as

$$\begin{aligned}
&ns(\eta, m) = \frac{1}{sn(\eta, m)}, \quad nc(\eta, m) = \frac{1}{cn(\eta, m)}, \quad nd(\eta, m) = \frac{1}{dn(\eta, m)},\\
&sc(\eta, m) = \frac{sn(\eta, m)}{cn(\eta, m)}, \quad sd(\eta, m) = \frac{sn(\eta, m)}{dn(\eta, m)}, \quad ds(\eta, m) = \frac{dn(\eta, m)}{sn(\eta, m)},\\
&cs(\eta, m) = \frac{cn(\eta, m)}{sn(\eta, m)}.
\end{aligned}$$

These Jacobian functions satisfy $sn^2\eta + cn^2\eta = 1$ and $dn^2\eta + m^2 sn^2\eta = 1$. We have

$$\begin{aligned}
&sn'\eta = cn\eta\, dn\eta, \quad cn'\eta = -sn\eta\, dn\eta, \quad dn'\eta = -m^2 sn\eta\, cn\eta, \quad ns'\eta = -cs\eta\, ds\eta,\\
&nc'\eta = sc\eta\, dc\eta, \quad nd'\eta = m^2 sd\eta\, cd\eta, \quad cs'(u) = -ns(u)ds(u),\\
&sc'(u) = nc(u)dc(u), \quad ds'(u) = -cs(u)\, ns(u), \quad sd'(u) = cd(u)\, nd(u).
\end{aligned}$$

We see that the degree of the function v given by (2.4) is $D(v) = N$, however the degree of v' is $D(v') = N + 1$.

3. Exact Solutions

In this subsection, we give an exact solution of the equation

$$u_t + \alpha u^2 u_x - \beta u_{xxx} = 0, \tag{3.1}$$

using the EJEFEM method.

Using the transformation $\eta = k(x+wt)$ and $u = u(x,t) = v(\eta)$, the equation (3.1) becomes

$$wv' + \alpha v^2 v' - \beta k^2 v^{(3)} = 0. \tag{3.2}$$

We integrate the equation (3.2) and we take the integral constant be equal to zero, we get

$$wv + \frac{\alpha}{3} v^3 - \beta k^2 v'' = 0. \tag{3.3}$$

Using (2.4) and balancing the degree of the non linear term v^3 with the degree of v'', we get $3N = N+2$ and so $N = 1$. We have

$$v(\eta) = a_0 + a_1 sn(\eta) + b_1 cn(\eta). \tag{3.4}$$

Therefore,

$$v'(\eta) = a_1 cn(\eta) dn(\eta) - b_1 sn(\eta) dn(\eta),$$

and

$$v''(\eta) = -a_1 sn(\eta) dn^2(\eta) - a_1 m^2 cn^2(\eta) sn(\eta) - b_1 cn(\eta) dn^2(\eta) + b_1 m^2 sn^2(\eta) cn(\eta).$$

Using the relationships of elliptic Jacobian functions, we obtain

$$\begin{aligned} v''(\eta) = &- a_1 sn(\eta) - a_1 m^2 sn(\eta) + 2a_1 m^2 sn^3(\eta) \\ &- b_1 cn(\eta) + 2b_1 m^2 cn(\eta) sn^2(\eta). \end{aligned} \tag{3.5}$$

Now,

$$v^2 = a_0^2 + b_1^2 + 2a_0 b_1 cn(\eta) + 2a_0 a_1 sn(\eta) + 2a_1 b_1 cn(\eta) sn(\eta) + a_1^2 sn^2(\eta) - b_1^2 sn^2(\eta) \tag{3.6}$$

and

$$\begin{aligned} v^3 = & a_0^3 + 3a_0 b_1^2 + 3a_0^2 b_1 cn(\eta) + b_1^3 cn(\eta) + 3a_0^2 a_1 sn(\eta) + 3a_1 b_1^2 sn(\eta) \\ & + 6a_0 a_1 b_1 cn(\eta) sn(\eta) + 3a_0 a_1^2 sn^2(\eta) - 3a_0 b_1^2 sn^2(\eta) + 3a_1^2 b_1 cn(\eta) sn^2(\eta) \\ & - b_1^3 cn(\eta) sn^2(\eta) + a_1^3 sn^3(\eta) - 3a_1 b_1^2 sn^3(\eta). \end{aligned} \tag{3.7}$$

Replacing v, v'' and v^3 in the equation (3.3) by their values using (3.4), (3.5) and (3.7) we get an equation of the form

$$\sum_{i=0}^{3} L_{i,0} sn^i(\eta) + L_{i,1} sn^i(\eta) cn(\eta) = 0, \tag{3.8}$$

where

$$L_{0,0} = a_0 w + (1/3)a_0^3\alpha + a_0 b_1^2\alpha,$$

$$L_{0,1} = b_1 w + a_0^2 b_1\alpha + (1/3)b_1^3\alpha + b_1 k^2\beta,$$

$$L_{1,0} = a_1 w + a_0^2 a_1\alpha + a_1 b_1^2\alpha + a_1 k^2\beta + a_1 k^2 m^2\beta,$$

$$L_{1,1} = 2a_0 a_1 b_1\alpha,$$

$$L_{2,0} = a_0 a_1^2\alpha - a_0 b_1^2\alpha,$$

$$L_{2,1} = a_1^2 b1\alpha - (1/3)b_1^3\alpha - 2b_1 k^2 m^2\beta,$$

$$L_{3,0} = (1/3)a_1^3\alpha - a_1 b_1^2\alpha - 2a_1 k^2 m^2\beta,$$

$$L_{3,1} = 0.$$

Such that the functions $1, cn, sn, sn\ cn, sn^2, sn^2\ cn, sn^3, sn^3\ cn$ are linearly independent, all the coefficients of the equation (3.7) are equal to zero. By using Wolfram Mathematica package, we get

$$\begin{aligned} &a_0 \to 0, a_1 \to \pm\frac{km\sqrt{6\beta}}{\sqrt{\alpha}}, \\ &b_1 \to 0, \\ &w \to -k^2(1+m^2)\beta. \end{aligned}$$

A family of solutions is given by

$$u(x,t) = a_1 sn(k(x+wt)). \tag{3.9}$$

As long as $m \to 1$, a second family of solutions is

$$u(x,t) = -k\sqrt{\frac{6\beta}{\alpha}}\tanh(k(x-x_0) - 2k^3\beta t). \tag{3.10}$$

and also,

$$u_1(x,t) = k\sqrt{\frac{6\beta}{\alpha}}\tanh(k(x-x_0) - 2k^3\beta t). \tag{3.11}$$

is a family of solutions of the equation (3.7).

The exact solution obtained in (3.10) is represented in Figure 1 with $\alpha = 1$, $\beta = 0.1$, $k = 2$, $T = 3$ and $x_0 = 2$ in the interval $[0, 10]$. In order to see the time changes for the exact solution, we consider $t = 0, 0.75, 1.5, 2.25$ and $t = 3$.

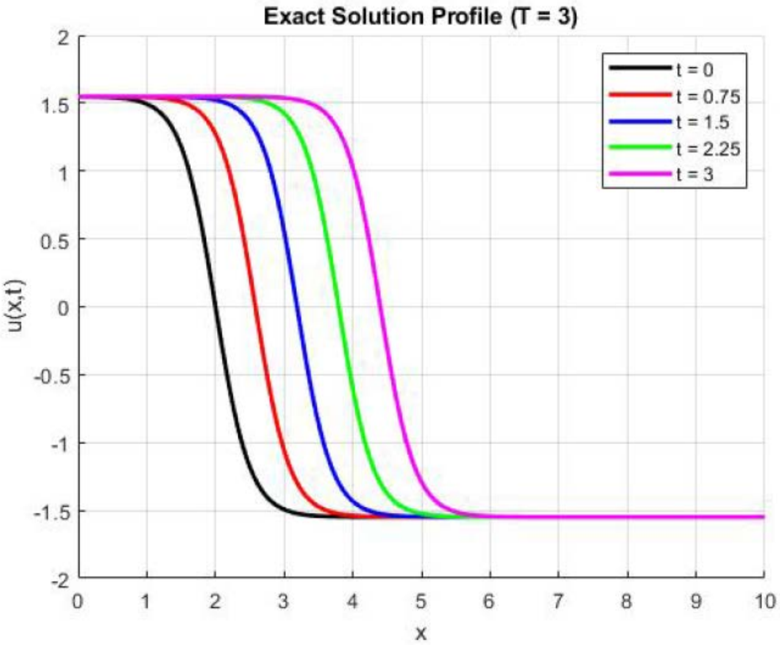

FIGURE 1. Illustration of the time evolution for $u(x,t)$ solution of the equation (3.1) with $\alpha = 1$, $\beta = 0.1$, $k = 2$ and $b = 10$

4. NUMERICAL RESULTS

In this section, we explore the numerical results of the equation

$$\begin{cases} u_t + \alpha u^2 u_x - \beta u_{xxx} = 0, \\ x \in [0,b] \text{ and } t \in [0,T]. \end{cases} \tag{4.1}$$

We use centered finite differences method to discretize the spatial derivatives, however the temporal derivative is kept continuous and as initial conditions we take

$$u(x,o) = -k\sqrt{\frac{6\beta}{\alpha}}\tanh(kx), \tag{4.2}$$

where k a positive constant. Such that the solution tends to zero at the end points of the physical domain, we take $u_x = 0$ and $u_{xxx} = 0$ as $x \to \pm\infty$. So, we can suppose that

$$u_x(0,t) = u_{xxx}(0,t) = 0 \text{ for all } t \in [0,T], \tag{4.3}$$

and

$$u_x(b,t) = u_{xxx}(b,t) = 0, \text{ for all } t \in [0,T] \tag{4.4}$$

as boundary conditions.

The finite differences method consists to replace the derivatives by some differences. For this, let $J > 0$ an integer, $\Delta x = \dfrac{b}{J}$ and $x_j = (j-1)\,\Delta x$, $j = 1, \cdots, J+1$. This means, we divided the domain $[0, b]$ to J equal subintervals.

For $N > 0$, we denote $\Delta t = \dfrac{T}{N}$ and $t^n = n\,\Delta t$, $n = 0, \cdots, N$.

From Taylor's formula, the centered difference for the space is given by

$$\frac{\partial u}{\partial x}(x,t) \approx \frac{u(x+\Delta x,t) - u(x-\Delta x,t)}{2\Delta x} \tag{4.5}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}. \tag{4.6}$$

So, when we discretize the equation (4.1), we get at each point (x_j, t):

$$\left\{ \begin{array}{c} u_{t,j} + \alpha u_j^2 (\dfrac{u_{j+1} - u_{j-1}}{2\Delta x}) - \beta (\dfrac{u_{xx,j+1} - u_{xx,j-1}}{2\Delta x}) = 0, \\ u_{xx,j} = \dfrac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2}. \end{array} \right. \tag{4.7}$$

Therefore,

$$\begin{aligned} u_t\,|_j^n &= -\alpha\,(u_j^{n+1})^2 \frac{(u_{j+1}^{n+1} - u_{j-1}^{n+1})}{2\Delta x} \\ &+ \frac{\beta}{2\Delta x}\,\frac{(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + u_j^{n+1}) - (u_j^{n+1} - 2u_{j-1}^{n+1} + u_{j-2}^{n+1})}{(\Delta x)^2}. \end{aligned} \tag{4.8}$$

Denote $r_1 = \dfrac{1}{2\Delta x}$ and $r_2 = \dfrac{1}{2(\Delta x)^3}$. The equation (4.8) gives

$$u_t\,|_j^n = -\alpha\, r_1\,(u_j^{n+1})^2 (u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \beta\, r_2 (u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}), \tag{4.9}$$

$$\forall\, 3 \le j \le J-1,\ 0 \le n \le N-1.$$

Approximating (4.2) directly gives

$$u_j^0 = f_j = -k\sqrt{\frac{6\beta}{\alpha}}\,\tanh(kx_j), \quad \forall\, 1 \le j \le J+1. \tag{4.10}$$

Approximating (4.3) by central difference, we have

$$\frac{u(0+\Delta x,t) - u(0-\Delta x,t)}{2\Delta x} = 0 \quad \text{and} \quad \frac{u_{xx}(0+\Delta x,t) - u_{xx}(0-\Delta x,t)}{2\Delta x} = 0$$

and so,

$$u_0^n = u_2^n \quad \text{and} \quad u_{xx,0}^n = u_{xx,2}^n, \quad \forall\, 0 \le n \le N. \tag{4.11}$$

From (4.11), we get
$$\frac{u_1^n - 2u_0^n + u_{-1}^n}{(\Delta x)^2} = \frac{u_3^n - 2u_2^n + u_1^n}{(\Delta x)^2}.$$

Therefore,

$$u_0^n = u_2^n \quad \text{and} \quad u_{-1}^n = u_3^n. \tag{4.12}$$

Next, approximating the right boundary condition (4.4) by central difference, we have
$$\frac{u(b+\Delta x, t) - u(b-\Delta x, t)}{2\Delta x} = 0$$

and
$$fracu_{xx}(b+\Delta x, t) - u_{xx}(b-\Delta x, t)2\Delta x = 0.$$

That is,

$$u_{J+2}^n = u_J^n \quad \text{and} \quad u_{xx,J+2}^n = u_{xx,J}^n. \tag{4.13}$$

Hence,
$$\frac{u_{J+3}^n - 2u_{J+2}^n + u_{J+1}^n}{(\Delta x)^2} = \frac{u_{J+1}^n - 2u_J^n + u_{J-1}^n}{(\Delta x)^2}.$$

Using (4.13) again, we obtain

$$u_{J+2}^n = u_J^n \quad \text{and} \quad u_{J+3}^n = u_{J-1}^n. \tag{4.14}$$

Therefore the scheme of the problem (4.1) is given by

$$u_t\,|_j^n = -\alpha\, r_1\,(u_j^{n+1})^2(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \beta\, r_2(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}), \tag{4.15}$$

for all $3 \le j \le J-1$ and $0 \le n \le N-1$ where $r_1 = \dfrac{1}{2\Delta x}$ and $r_2 = \dfrac{1}{2(\Delta x)^3}$ and from the boundary conditions, we obtain

$$\begin{cases} u_0^n = u_2^n, & u_{-1}^n = u_3^n, \\ u_{J+2}^n = u_J^n, & u_{J+3}^n = u_{J-1}^n. \end{cases} \tag{4.16}$$

Figures 2 and 3 show the exact and numerical solutions of the implementation of the spatial finite difference method with $\alpha = 1$, $\beta = 0.1$, $k = 2$, $T = 3$, $b = 10$ and $x_0 = 2$. Figure 2 represent the numerical solution vs the exact one at different times in the time domain $[0, 3]$ and Figure 3 illustrate the two solutions in 3D. In the figures, we remark the coherence of the two solutions.

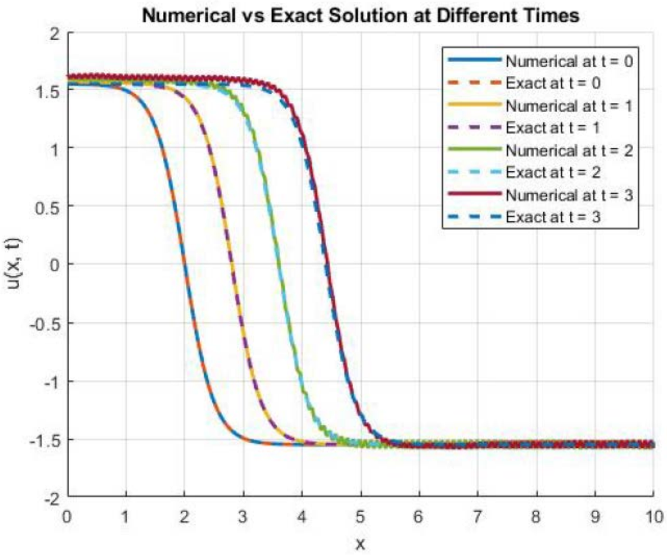

FIGURE 2. Comparison of the two solutions of the equation (3.1) with $\alpha = 1$, $\beta = 0.1$, $k = 2$ and $b = 10$ and for different times

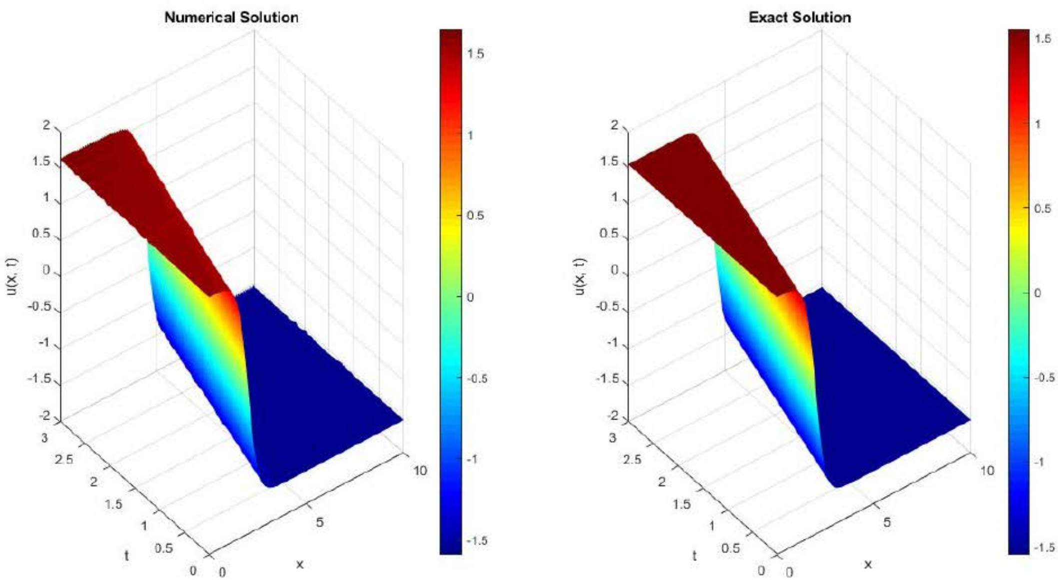

FIGURE 3. Illustration of the two solutions, the exact and the numerical, of the equation (3.1) with $\alpha = 1$, $\beta = 0.1$, $k = 2$ and $b = 10$ in $3D$

5. Von Neumann Stability Analysis

In order to use the the von Neumann analysis to study the stability of the scheme (4.11), we consider the following linear approach of the equation (1.1).

$$u_t + su_x - \beta u_{xxx} = 0, \tag{5.1}$$

where $s = \alpha u^2$. We set

$$u_j^n = \lambda^n e^{ikj\Delta x}, \text{ where } \lambda = \lambda(k).$$

We have

$$\begin{aligned}
u_j^{n+1} &= \lambda^{n+1} e^{ikj\Delta x} = \lambda u_j^n;\\
u_{j+1}^n &= \lambda^n e^{ik(j+1)\Delta x} = e^{ik\Delta x} u_j^n;\\
u_{j+2}^n &= \lambda^n e^{ik(j+2)\Delta x} = e^{2ik\Delta x} u_j^n;\\
u_{j-1}^n &= \lambda^n e^{ik(j-1)\Delta x} = e^{-ik\Delta x} u_j^n;\\
u_{j-2}^n &= \lambda^n e^{ik(j-2)\Delta x} = e^{-2ik\Delta x} u_j^n.
\end{aligned}$$

The scheme of the equation (5.1) at a grid point (x_j, t^n) and using backward finite difference for the time derivative is

$$u_j^n = u_j^{n+1} - s\, r_1 (u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \beta\, r_2\, (u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}). \tag{5.2}$$

Therefore,

$$\begin{aligned}
u_j^n = \lambda\Big(& u_j^n - s\, r_1 (e^{ik\Delta x} - e^{-ik\Delta x}) u_j^n + \beta\, r_2 (e^{2ik\Delta x} - e^{-2ik\Delta x}) u_j^n\\
& - 2\beta\, r_2 (e^{ik\Delta x} - e^{-ik\Delta x}) u_j^n \Big).
\end{aligned}$$

So, we get

$$1 = \lambda\Big(1 - 2s\, r_1\, i \sin(k\Delta x) + 2\beta\, r_2\, i \sin(2k\Delta x) - 4\beta\, r_2\, i \sin(k\Delta x)\Big).$$

Since $r_1 = \dfrac{\Delta t}{2\Delta x}$ and $r_2 = \dfrac{\Delta t}{2(\Delta x)^3}$, we obtain

$$1 = \lambda\Big(1 - s\frac{\Delta t}{\Delta x}\, i \sin(k\Delta x) + \beta \frac{\Delta t}{(\Delta x)^3}\, i \sin(2k\Delta x) - 2\beta \frac{\Delta t}{(\Delta x)^3}\, i \sin(k\Delta x)\Big).$$

That is,

$$1 = \lambda(1 - i\,\delta) \tag{5.3}$$

where $\delta = s\frac{\Delta t}{\Delta x}\sin(k\Delta x)+\beta\frac{\Delta t}{(\Delta x)^3}\sin(2k\Delta x)-2\beta\frac{\Delta t}{(\Delta x)^3}\sin(k\Delta x)$. From the equation (5.3), we get

$$|\lambda|^2 = \frac{1}{1+\delta^2} \leq 1.$$

We deduce that the condition of the stability is unconditionally satisfied.

6. Error analysis

In order to confirm the accuracy of the numerical scheme (5.2), we study the truncation error by Taylor's expansions method. First let

$$e_j^{n+1} = u_j^{n+1} - u(x_j, t^{n+1}) \tag{6.1}$$

the error, where u_j^{n+1} the approximative solution and $u(x_j, t^{n+1})$ the exact one on the grid points.

Second, we write (5.2) in the following way

$$\frac{u_j^n - u_j^{n+1}}{\Delta t} = -\frac{s}{2\Delta x}\Big(u_{j+1}^{n+1} - u_{j-1}^{n+1}\Big) + \frac{\beta}{2(\Delta x)^3}\Big(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}\Big). \tag{6.2}$$

We set T_j^{n+1} the truncation error. It represent the difference between the two sides of the equation (6.2), when we replaced u_j^{n+1} by $u(x_j, t^{n+1})$. This means T_j^{n+1} quantifies the difference between the exact equation and its numerical approximation and we have

$$\begin{aligned} T_j^{n+1} =& \frac{u(x_j, t^n) - u(x_j, t^{n+1})}{\Delta t} + \frac{s}{2\Delta x}\Big(u(x_{j+1}, t^{n+1}) - u(x_{j-1}, t^{n+1})\Big) \\ &- \frac{\beta}{2(\Delta x)^3}\Big(u(x_{j+2}, t^{n+1}) - 2u(x_{j+1}, t^{n+1}) + 2u(x_{j-1}, t^{n+1}) - u(x_{j-2}, t^{n+1})\Big). \end{aligned} \tag{6.3}$$

Now, from (6.1), we have $u_j^{n+1} = e_j^{n+1} + u(x_j, t^{n+1})$. When we substitute u_j^{n+1} into the numerical scheme (6.2) and use (6.3), we get the error equation

$$\frac{e_j^n - e_j^{n+1}}{\Delta t} = T_j^{n+1} + \frac{s}{2\Delta x}(e_{j+1}^{n+1} - e_{j-1}^{n+1}) - \frac{\beta}{2(\Delta x)^3}(e_{j+2}^{n+1} - 2e_{j+1}^{n+1} + 2e_{j-1}^{n+1} - e_{j-2}^{n+1}). \tag{6.4}$$

But, if we denote

$$\delta_x v(x,t) = v(x+\frac{1}{2}\Delta x, t) - v(x-\frac{1}{2}\Delta x, t)$$

for a function $v(x, y)$, we have

$$\delta_x^2 v(x,t) = \Big(v(x+\Delta x, t) - v(x,t)\Big) - \Big(v(x,t) - v(x-\Delta x, t)\Big).$$

Therefore,

$$\delta_x^2 v(x,t) = v(x+\Delta x, t) - 2v(x,t) + v(x-\Delta x, t).$$

From (6.4), we get

$$\frac{e_j^n - e_j^{n+1}}{\Delta t} = T_j^{n+1} + \frac{s}{2\Delta x}(e_{j+1}^{n+1} - e_{j-1}^{n+1}) - \frac{\beta}{2(\Delta x)^3}\delta_x^2(e_{j+1}^{n+1} - e_{j-1}^{n+1}). \tag{6.5}$$

6.1. **Truncation Error Analysis.** Consider the truncation error given by the equation (6.3) for the linearized equation $u_t + su_x - \beta u_{xxx} = 0$.

First, for the numerical approximation for the time derivative $\dfrac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t}$, performing a Taylor series expansion around t^n, we get

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t\; u_t(x_j, t^n) + \frac{(\Delta t)^2}{2} u_{tt}(x_j, t^n) + O(\Delta t^3).$$

Then, we obtain

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = u_t(x_j, t^n) + \frac{\Delta t}{2} u_{tt}(x_j, t^n) + O(\Delta t^2). \tag{6.6}$$

Thus, the time error is of order $O(\Delta t)$.

Second, for the spatial error of u_x, we have the approximation

$$\frac{u(x_{j+1}, t^{n+1}) - u(x_{j-1}, t^{n+1})}{2\Delta x}.$$

By expanding each term in a Taylor series around x_j, we find

$$u(x_{j+1}, t^{n+1}) = u(x_j, t^{n+1}) + \Delta x\; u_x(x_j, t^{n+1}) + \frac{(\Delta x)^2}{2} u_{xx}(x_j, t^{n+1}) + O(\Delta x^3)$$

and

$$u(x_{j-1}, t^{n+1}) = u(x_j, t^{n+1}) - \Delta x\; u_x(x_j, t^{n+1}) + \frac{(\Delta x)^2}{2} u_{xx}(x_j, t^{n+1}) + O(\Delta x^3)\cdot$$

Hence,

$$\frac{u(x_{j+1}, t^{n+1}) - u(x_{j-1}, t^{n+1})}{2\Delta x} = u_x(x_j, t^{n+1}) + O(\Delta x^2). \tag{6.7}$$

Therefore, the spatial error for u_x is of order $O(\Delta x^2)$.

Finally, the approximation of the third spatial derivative u_{xxx} is given by

$$\frac{u(x_{j+2}, t^{n+1}) - 2u(x_{j+1}, t^{n+1}) + 2u(x_{j-1}, t^{n+1}) - u(x_{j-2}, t^{n+1})}{2(\Delta x)^3}.$$

Since

$$\begin{aligned}u(x_{j+2},t^{n+1}) &= u(x_j,t^{n+1}) + 2\Delta x\ u_x(x_j,t^{n+1}) + 2(\Delta x)^2 u_{xx}(x_j,t^{n+1})\\ &\quad + \frac{4(\Delta x)^3}{6} u_{xxx}(x_j,t^{n+1}) + O(\Delta x^4),\\ u(x_{j-2},t^{n+1}) &= u(x_j,t^{n+1}) - 2\Delta x\ u_x(x_j,t^{n+1}) + 2(\Delta x)^2 u_{xx}(x_j,t^{n+1})\\ &\quad - \frac{4(\Delta x)^3}{6} u_{xxx}(x_j,t^{n+1}) + O(\Delta x^4),\end{aligned}$$

we get

$$\begin{aligned}&u(x_{j+2},t^{n+1}) - 2u(x_{j+1},t^{n+1}) + 2u(x_{j-1},t^{n+1}) - u(x_{j-2},t^{n+1})\\ &= 2(\Delta x)^3 u_{xxx}(x_j,t^{n+1}) + O(\Delta x^5).\end{aligned} \tag{6.8}$$

Therefore, the error for u_{xxx} is of order $O(\Delta x^2)$.

We conclude that the total truncation error satisfies $T_j^{n+1} = O(\Delta t) + O(\Delta x^2)$.

6.2. **Relative error with L_2 norm.** In order to confirm the stability of our approximative solution for the the nonlinear KdV equation of second order nonlinearity given by (1.1), we will calculate the relative error with the L_2 norm and the (CPU) time which indicates a computer responsible for executing instructions and performing calculations. We consider $\alpha = 1$, $\beta = 0.1$, $k = 2$ and $T = 3$. We get the following table.

Δ_x	Erreur Relative (L_2)	Temps CPU (s)
0.1500	1.0211e-01	0.1351
0.0750	4.0758e-02	0.6639
0.0375	2.9308e-02	0.4900
0.0187	1.1401e-01	3.2443
0.0094	5.6102e-02	12.4567

Through the above table, we remark that the error approaches zero whenever the value of Δx, as spatial step size, is small. In addition, whenever Δx decreases, the error generally decreases, indicating that the numerical method converges toward the exact solution. However, for very small Δx (e.g., 0.0187), the error slightly increases, which could be due to numerical instability or accumulated rounding errors. The CPU time increases significantly as Δx becomes smaller, reflecting the higher computational cost of finer grids. These results demonstrate the trade-off between accuracy and computational efficiency in numerical simulations.

7. Conclusion

In this paper, we use the extended Jacobian elliptic function expansion method to solve some nonlinear KdV equations with nonlinearity of degree 2. After that, having exact solutions, we solve these equations numerically by using the finite differences method for the space variable and keeping the time continuous. We write the scheme taking in consideration the initial condition and boundary conditions of type Neumann. We compare the two solutions, exact and numerical, for the case $\alpha = 1$, $\beta = 0.1$, $k = 2$ and $T = 3$, in different times (Fig. 2) and as functions on (x, t) (Fig. 3). For more numerical investigation, we calculate the traduction errors which is of order $O(\Delta t) + O(\Delta x^2)$. Finally, we calculate the L_2-relative errors and compare the errors with the values of Δx.

We see and prove that the semi-finite differences method is suitable to this type of nonlinear partial differential equations even if the the nonlinearity is of order 2.

8. Conflict of interest

The author declares that there is no conflict of interest.

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