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REVISITING TWO VARIANTS OF THE HARDY-HILBERT INTEGRAL INEQUALITY

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ABSTRACT. The Hardy-Hilbert integral inequality is fundamental to mathematical analysis. It has inspired a vast body of research devoted to its generalizations and refinements. This article sheds new light on the subject by demonstrating that certain variants of the inequality possess a self-extending property. This property enables us to derive well-known results from existing literature. Two distinct formulations are developed within different analytical frameworks. Reproducible proofs are provided to illustrate the underlying mechanism, which is based on successive power-type changes of variables.

1. INTRODUCTION

The Hardy-Hilbert integral inequality was popularized in [4]. It has attracted significant attention due to its fundamental role in analysis and the many ways in which it has been extended. A statement of the original form is given below. Let $p > 1$ and q such that $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

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Then, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \le \frac{\pi}{\sin(\pi/p)}\left(\int_0^{+\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}.$$

The constant factor $\pi/\sin(\pi/p)$ is the best possible. There is extensive literature on the Hardy-Hilbert integral inequality and its numerous variants. For a comprehensive overview of the subject, we recommend the book [9], as well as to the key contributions [3–8, 10] and the recent advances reported in [1, 2].

In this article, we demonstrate that certain variants of the Hardy-Hilbert integral inequality have a self-extending property, through which well-known results from the existing literature can be obtained. This phenomenon is illustrated by two distinct formulations developed within the frameworks of [8] and [10]. The central proof mechanism involves successive power-type changes of variables. Every proof is presented in full and can be reproduced in its entirety. In a sense, we revisit the main results in [8] and [10] via a new approach.

The remainder of the article is as follows: The first and second formulations are presented in Sections 2 and 3, respectively. A conclusion is provided in Section 4.

2. First formulation

In addition to the Hardy-Hilbert integral inequality, a famous result in [4] involves the logarithmic function. It is formally stated below. Let $p > 1$ and q such that $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

Then, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{\ln(x/y)}{x-y}f(x)g(y)dxdy\\ &\le \left(\frac{\pi}{\sin(\pi/p)}\right)^2\left(\int_0^{+\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}.\end{aligned} \tag{2.1}$$

The constant factor $(\pi/\sin(\pi/p))^2$ is the best possible. There are numerous variants of this inequality. In particular, the main result in [8] is a one-parameter generalization. A possible statement is given below. Let $\lambda > 0$, $p > 1$ and q such that $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} x^{(p-1)(1-\lambda)} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{(q-1)(1-\lambda)} g^q(y)dy < +\infty.$$

Then, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)g(y)dxdy \le \left(\frac{\pi}{\lambda \sin(\pi/p)}\right)^2$$
$$\cdot \left(\int_0^{+\infty} x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q}. \tag{2.2}$$

The constant factor $(\pi/(\lambda \sin(\pi/p)))^2$ is the best possible.

Our first contribution is formulated in the theorem below.

Theorem 2.1. *The inequality in Equation* (2.1) *implies the inequality in Equation* (2.2).

Proof. The proof mechanism involves successive power-type changes of variables. Starting with the double integral in Equation (2.2), making the changes of variables $u = x^\lambda$ and $v = y^\lambda$, so that $x = u^{1/\lambda}$ and $y = v^{1/\lambda}$ and $dx = (1/\lambda)u^{1/\lambda-1}du$ and $dy = (1/\lambda)v^{1/\lambda-1}dv$, and using $\ln(u^{1/\lambda}/v^{1/\lambda}) = \ln\left((u/v)^{1/\lambda}\right) = (1/\lambda)\ln(u/v)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{\ln(u^{1/\lambda}/v^{1/\lambda})}{u - v} f(u^{1/\lambda})g(v^{1/\lambda}) \tag{2.3}$$
$$\cdot \left(\frac{1}{\lambda}u^{1/\lambda-1}du\right)\left(\frac{1}{\lambda}v^{1/\lambda-1}dv\right) \tag{2.4}$$
$$= \frac{1}{\lambda^3}\int_0^{+\infty}\int_0^{+\infty} \frac{\ln(u/v)}{u - v} f_\star(u)g_\star(v)dudv,$$

where

$$f_\star(u) = u^{1/\lambda-1} f(u^{1/\lambda}), \quad g_\star(v) = v^{1/\lambda-1} g(v^{1/\lambda}).$$

Applying the inequality in Equation (2.1) to $f_\star$ and $g_\star$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\ln(u/v)}{u - v} f_\star(u)g_\star(v)dudv$$

$$
\begin{aligned}
&\le \left(\frac{\pi}{\sin(\pi/p)}\right)^2 \left(\int_0^{+\infty} f_\star^p(u)du\right)^{1/p} \left(\int_0^{+\infty} g_\star^q(v)dv\right)^{1/q} \\
&= \left(\frac{\pi}{\sin(\pi/p)}\right)^2 \left(\int_0^{+\infty} u^{(1/\lambda-1)p} f^p(u^{1/\lambda})du\right)^{1/p} \\
&\quad \cdot \left(\int_0^{+\infty} v^{(1/\lambda-1)q} g^q(v^{1/\lambda})dv\right)^{1/q}. \qquad (2.5)
\end{aligned}
$$

Making back the changes of variables $u = x^\lambda$ and $v = y^\lambda$, so that $du = \lambda x^{\lambda-1}dx$ and $dv = \lambda y^{\lambda-1}dy$, and using $1/p + 1/q = 1$, we get

$$
\begin{aligned}
&\left(\int_0^{+\infty} u^{(1/\lambda-1)p} f^p(u^{1/\lambda})du\right)^{1/p} \left(\int_0^{+\infty} v^{(1/\lambda-1)q} g^q(v^{1/\lambda})dv\right)^{1/q} \\
&= \left(\int_0^{+\infty} x^{\lambda(1/\lambda-1)p} f^p(x)(\lambda x^{\lambda-1}dx)\right)^{1/p} \qquad (2.6) \\
&\quad \cdot \left(\int_0^{+\infty} y^{\lambda(1/\lambda-1)q} g^q(y)(\lambda y^{\lambda-1}dy)\right)^{1/q} \\
&= \lambda \left(\int_0^{+\infty} x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q}.
\end{aligned}
$$

Combining Equations (2.3), (2.5) and (2.6), we derive

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)g(y)dxdy \\
&\le \frac{1}{\lambda^3}\left(\frac{\pi}{\sin(\pi/p)}\right)^2 \lambda \left(\int_0^{+\infty} x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q} \\
&= \left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2 \left(\int_0^{+\infty} x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\lambda)} g^q(y)dy\right)^{1/q}.
\end{aligned}
$$

We obtain the inequality in Equation (2.2). Therefore, the inequality in Equation (2.1) implies the inequality in Equation (2.2), completing the proof. □

This theorem demonstrates the self-extending property of the inequality in Equation (2.1), enabling it to derive a previously established theorem from the literature. Alternatively, we revisited the inequality in Equation (2.2) using Equation (2.1) as a baseline.

3. Second formulation

The main result in [5] is formally stated below. Let $a, b > 0$ with $a \neq b$, $p > 1$ and q such that $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} x^{-p-1} f^p(x) dx < +\infty, \quad \int_0^{+\infty} y^{-q-1} g^q(y) dy < +\infty.$$

Then, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x + ay)^2 (x + by)^2} f(x) g(y) dx dy$$
$$\leq \Upsilon \left(\int_0^{+\infty} x^{-p-1} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} y^{-q-1} g^q(y) dy \right)^{1/q}, \tag{3.1}$$

where

$$\Upsilon = \frac{a + b}{(b - a)^2} \left(\frac{1}{b - a} \ln \left(\frac{b}{a} \right) - \frac{2}{a + b} \right).$$

The constant factor Υ is the best possible. Despite the possibilities offered by the adjustable parameters a and b, this result is little known. One of the advances is the one-parameter generalization of the inequality presented in [10]. We provide its statement below. Let $\lambda > 0$, $a, b > 0$ with $a \neq b$, $p > 1$ and q such that $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} x^{p(1-2\lambda)-1} f^p(x) dx < +\infty, \quad \int_0^{+\infty} y^{q(1-2\lambda)-1} g^q(y) dy < +\infty.$$

Then, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x^\lambda + ay^\lambda)^2 (x^\lambda + by^\lambda)^2} f(x) g(y) dx dy$$
$$\leq \Xi \left(\int_0^{+\infty} x^{p(1-2\lambda)-1} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} y^{q(1-2\lambda)-1} g^q(y) dy \right)^{1/q}, \tag{3.2}$$

where

$$\Xi = \frac{1}{\lambda} \Upsilon = \frac{a + b}{\lambda (b - a)^2} \left(\frac{1}{b - a} \ln \left(\frac{b}{a} \right) - \frac{2}{a + b} \right).$$

The constant factor Ξ is the best possible.

Our second contribution is formulated in the theorem below.

Theorem 3.1. *The inequality in Equation* (3.1) *implies the inequality in Equation* (3.2).

Proof. As in the proof of Theorem 2.1, we consider successive power-type changes of variables. Starting with the double integral in Equation (3.2), and making the changes of variables $u = x^\lambda$ and $v = y^\lambda$, so that $x = u^{1/\lambda}$ and $y = v^{1/\lambda}$ and $dx = (1/\lambda)u^{1/\lambda-1}du$ and $dy = (1/\lambda)v^{1/\lambda-1}dv$, we have

$$\begin{aligned}(3.3)\qquad &\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x^\lambda+ay^\lambda)^2(x^\lambda+by^\lambda)^2}f(x)g(y)dxdy\\ &=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(u+av)^2(u+bv)^2}f(u^{1/\lambda})g(v^{1/\lambda})\\ &\cdot\left(\frac{1}{\lambda}u^{1/\lambda-1}du\right)\left(\frac{1}{\lambda}v^{1/\lambda-1}dv\right)\\ &=\frac{1}{\lambda^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(u+av)^2(u+bv)^2}f_\star(u)g_\star(v)dudv,\end{aligned}$$

where

$$f_\star(u) = u^{1/\lambda-1}f(u^{1/\lambda}),\quad g_\star(v) = v^{1/\lambda-1}g(v^{1/\lambda}).$$

Applying the inequality in Equation (3.1) to $f_\star$ and $g_\star$, we obtain

$$\begin{aligned}(3.4)\qquad &\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(u+av)^2(u+bv)^2}f_\star(u)g_\star(v)dudv\\ &\leq \Upsilon\left(\int_0^{+\infty}u^{-p-1}f_\star^p(u)du\right)^{1/p}\left(\int_0^{+\infty}v^{-q-1}g_\star^q(v)dv\right)^{1/q}\\ &=\Upsilon\left(\int_0^{+\infty}u^{-p-1}u^{(1/\lambda-1)p}f^p(u^{1/\lambda})du\right)^{1/p}\\ &\cdot\left(\int_0^{+\infty}v^{-q-1}v^{(1/\lambda-1)q}g^q(v^{1/\lambda})dv\right)^{1/q}\\ &=\Upsilon\left(\int_0^{+\infty}u^{(1/\lambda-2)p-1}f^p(u^{1/\lambda})du\right)^{1/p}\left(\int_0^{+\infty}v^{(1/\lambda-2)q-1}g^q(v^{1/\lambda})dv\right)^{1/q}.\end{aligned}$$

Making back the changes of variables $u = x^\lambda$ and $v = y^\lambda$, so that $du = \lambda x^{\lambda-1}dx$ and $dv = \lambda y^{\lambda-1}dy$, and using $1/p + 1/q = 1$, we get

$$\begin{aligned}&\left(\int_0^{+\infty}u^{(1/\lambda-2)p-1}f^p(u^{1/\lambda})du\right)^{1/p}\left(\int_0^{+\infty}v^{(1/\lambda-2)q-1}g^q(v^{1/\lambda})dv\right)^{1/q}\\ &=\left(\int_0^{+\infty}x^{\lambda(1/\lambda-2)p-\lambda}f^p(x)(\lambda x^{\lambda-1}dx)\right)^{1/p}\left(\int_0^{+\infty}y^{\lambda(1/\lambda-2)q-\lambda}g^q(y)(\lambda y^{\lambda-1}dy)\right)^{1/q}\end{aligned}$$

$$= \lambda \left(\int_0^{+\infty} x^{p(1-2\lambda)-1} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} y^{q(1-2\lambda)-1} g^q(y) dy \right)^{1/q}. \tag{3.5}$$

Combining Equations (3.3), (3.4) and (3.5), we derive

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x^\lambda + ay^\lambda)^2 (x^\lambda + by^\lambda)^2} f(x) g(y) dx dy \\ &\leq \frac{1}{\lambda^2} \Upsilon \lambda \left(\int_0^{+\infty} x^{p(1-2\lambda)-1} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} y^{q(1-2\lambda)-1} g^q(y) dy \right)^{1/q} \\ &= \Xi \left(\int_0^{+\infty} x^{p(1-2\lambda)-1} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} y^{q(1-2\lambda)-1} g^q(y) dy \right)^{1/q}. \end{aligned}$$

We obtain the inequality in Equation (3.2). Therefore, the inequality in Equation (3.1) implies the inequality in Equation (3.2), completing the proof. □

This theorem demonstrates that the inequality in Equation (3.1) possesses a self-extending property, allowing it to recover a previously established theorem in the literature. Alternatively, we revisited the inequality in Equation (3.2) using Equation (3.1) as a baseline.

4. Conclusion

This study has demonstrated that certain variants of the Hardy-Hilbert integral inequality have a self-extending property, which enables classical results to be recovered through successive power-type changes of variables. This approach clarifies the underlying structural unity of several known inequalities. It also provides a flexible framework for further generalisations. Future work could involve exploring extensions to multidimensional settings, inequalities involving weighted kernels and applications to functional and operator analysis.

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