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THE SPECTRAL ANALYSIS OF NEWLY DEFINED NORLUND MATRICES: THE EIGENVALUES AS AN OPERATOR ON C AND C_0 SPACES

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ABSTRACT. This paper investigates the spectral analysis of newly defined Norlund infinite matrix: the eigenvalues as an operator on the sequence spaces c_0 and c. The newly defined Norlund matrix, a conception of Cesaro matrix, has been greatly studied in the context of summability theory. However, its spectral analysis on the sequence space c_0 and c stay mostly undiscovered. This paper explored and presented the spectral analysis of newly defined Norlund Infinite matrix: the eigenvalues of a Norlund Infinite matrix as an operator on the sequence space c_0 and c. Through the eigenvalue problem $Zx = \lambda x$. The findings at the end of the research are $Z \in B(c_0)$ has no eigenvalues and $Z \in B(c)$ has the set 1 and their spectra presented.

1. INTRODUCTION

Mathematicians have been motivated by the study of infinite matrices and their properties for over a century. Norlund matrices are widely used in the field of functional analysis, operator theory, and numerical analysis. The Norlund infinite matrix, first described by Niels Erik Norlund in 1920, is a significant expansion of the Cesaro matrix, with far- reaching implications for various disciplines of mathematics, including approximation theory, functional analysis, and number theory.

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A crucial aspect of understanding the characteristics of infinite matrices as an operator on sequence spaces lies in explaining their spectral analysis, particularly the eigenvalues, which serve as a fundamental determinant of their functional characteristics. However, the eigenvalues of a Norlund matrices as an operator on sequence spaces remain unexplored. Understanding the spectral properties of Norlund matrices is essential for grasping their beginning characteristics. Norlund matrices are structured as infinite matrices represented by:

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & \cdots \\ z_{21} & z_{22} & z_{23} & \cdots \\ z_{31} & z_{32} & z_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.1}$$

where the elements of (z_{ij}) are complex numbers. These matrices have been thoroughly investigated within the framework of sequence spaces, particularly the spaces c and c_0. The space c includes all convergent sequences, while c_0 encompasses sequences that converge to zero. The spectral analysis of Norlund matrices focuses on their eigenvalues and eigenvectors. While eigenvectors are non-zero vectors that produce an estimated interpretation of themselves when witnessing a direct transformation, eigenvalues are scalar amounts that show how much a direct transformation alters a vector. This paper aims to add further present some analysis on the theoretical structure surrounding spectral analysis of infinite matrices focusing on Norlund matrices as bounded linear operators on the sequence spaces c_0 and c.

2. Preliminaries

2.1. **Norlund Matrices.** A newly defined Norlund matrix $Z = (z_{nk})$ is generated from a sequence weight p_k and is defined by $Z_nk = \frac{P_{n-k}}{P_n}$, for $0 \leq k \leq n$.
In this study we consider the weight sequence $P_0 = P_1 = P_2 = m$ where m is a constant, that is $m = 1$, $P_k = 0$ for $k \geq 3$. Then the cumulative sum becomes,

$$Z_{nk} = \begin{cases} m, & n = 0 \\ 2m, & n = 1 \\ 3m, & n \geq 0 \end{cases}. \tag{2.1}$$

Equation (2.1) defines a Norlund matrix $Z = (Z_n k)$ by means of:

$$Z_{nk} = \frac{P_{n-k}}{P_n} \quad \text{for} \quad 0 \leq k \leq n \tag{2.2}$$

Hence, the matrix Z is given as

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.3}$$

This is tri-diagonal average matrix starting from row 2.

2.2. **Sequence Spaces.** Sequence spaces are mathematical structures that consists of sets of sequences with certain characteristics and operations. Sequence spaces, such as c_0 play a crucial role in functional analysis. The space c_0 is equipped with norms that assist the exploration of convergence and boundedness. It consists of sequences that converge to zero being it real or complex numbers. For example, having these sequences $x_n = \frac{1}{n}$ and $x_n = \frac{(-1)^n}{n}$ belong to c_0 where n is a natural number. Also, the space c comprises of all convergent sequence of complex or real values. The sequence $x_n = \frac{n+1}{n}$ and $x_n = 2 + \frac{1}{n}$ where n is a natural number, belong to c. Both of these spaces are Banach spaces under the supremum norm $\|x\|_\infty = \sup_n |x_n|$.

2.3. **Eigenvalues.** They are the unique real numbers that is related to the set of linear equations in matrix equation. Eigenvalues associated with matrices and linear transformation, are fundamental concepts in functional analysis and linear algebra. They offer significant new viewpoint on how these changes depict the behaviour of systems. Let us explore the problem obtaining vectors (vector columns) $x (x \neq 0)$ and numbers λ (real or complex) given the square matrix Z such that:

$$Zx = \lambda x. \tag{2.4}$$

The non-zero vector x is referred to as the eigenvectors agreeing to the eigenvalues λ and the number λ is referred to as the eigenvalues of the matrix Z. This problem is known as the eigenvalue problem. With I being the identity matrix, we may find the eigenvalues by noting that $Zx = \lambda Ix$. Hence, equation (2.4) can be rewritten as:

$$Zx = \lambda Ix, \tag{2.5}$$

$$Zx - \lambda Ix = 0. \tag{2.6}$$

Thus

$$x(Z - \lambda I) = 0. \tag{2.7}$$

The non-trivial solution to equation (2.7), that actually reflects the linear system, is found if and only if the matrix $Z - \lambda I$ of this system is unique. This is the instance for which

$$\det(Z - \lambda I) = 0. \tag{2.8}$$

Hence, the equation for determining eigenvalues λ is obtained and equation (2.8) is the characteristics equation.

2.4. **Operator Theory.** This is a branch of functional analysis that deals with the exploration of linear operators on vector spaces. It gives a vital structure for exploring and comprehending the features of linear transformations. The following are key concepts in operator theory.

i. Linear Operators.
This is a mapping between vector spaces that preserves the operations of vector addition and scalar multiplication. If X and Y are vector spaces over a field $\mathbb{C}$ or $\mathbb{R}$, the function $Z : X \to Y$ is a linear operator if:
   a. Additivity:

$$Z(x_1 + x_2) = Z(x_1) + Z(x_2) \quad \text{for} \quad x_1 \quad \text{and} \quad x_2 \in X. \tag{2.9}$$

   b. Homogeneity: $Z(\lambda x_1) = \lambda Z(x_1)$ for all $x_1 \in X$ and scalars λ. These properties collectively give the complete definition of linearity:

$$Z(\lambda x_1 + \mu x_2) = \lambda Z(x_1) + \mu Z(x_2) \quad \forall x_1, x_2 \in X \quad \text{and scalars} \quad \mu, \lambda. \tag{2.10}$$

   The context for bounded linear operators on normed spaces, as well as spectral analysis, is established in Banach's foundational work on linear operations ([1]).

ii. Boundedness: A linear operator $Z : Y \to Y$ between normed vector spaces X and Y is bounded if there exists a constant $M > 0$ such that:

$$\|Zx\| \leq M\|x\|, \quad \forall x \in X. \tag{2.11}$$

All bounded linear operators between normed spaces are continuous. On the other hand, all continuous linear operators are bounded ([6]); ([9]).

2.5. **Banach Spaces.** The concept of a Banach space is basic to the study of modern functional analysis named after the Polish mathematician Stefan Banach. It generalizes the familiar notion of Euclidean space to possibly infinite-dimensional settings. The development of Banach space theory provides a unified framework for discussing continuity, convergence and boundedness of operators in abstract vector spaces.

Definition 2.1. *Let X be a vector space over the field $\mathbb{C}$ or $\mathbb{R}$, equipped with a norm $\|\cdot\|$. Then $(X, \|\cdot\|)$ is called a Banach space if it is complete with respect to the metric induced by the norm. That is, every Cauchy sequence $\{x_n\} \subset X$ converges to a point $x \in X$ such that*

$$\lim_{n\to\infty} \|x_n - X\| = 0. \tag{2.12}$$

The completeness property ensures that Banach spaces are closed under the limit operations, making them a perfect setting for various analytical constructions and fixed-point theorems. Many foundational results in analysis, such as the Uniform Boundedness Principle, the Hahn-Banach Theorem and the Banach Fixed Point Theorem are within the context of Banach spaces ([5]); ([3]). Two classical examples examples of Banach spaces that are relevant to this study are sequence spaces c_0 and c. Both spaces are provided with the supremum norm defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|. \tag{2.13}$$

With this norm, both c_0 and c are complete normed vector spaces and therefore Banach spaces ([11]);([5]). Their completeness under the supremum norm is an important characteristic that allows for the application of diverse tools from functional analysis, such as compact operators and spectral theory. They are also important in the study of operator theory and matrix transformations. Primarily, operators defined by infinite matrices, for instance the Norlund matrix, act naturally on c_0 and c, and examining their behaviour needs an understanding of the linear and topological structure of these spaces ([6]) ([9]). The Banach space context ensures that limits of sequences and operator norms are well-defined, which is essential for spectral analysis in this thesis. Moreover, since both c_0 and c are subspaces of the ℓ^∞ of all bounded sequences, they acquire numerous topological functional characteristics that make them appropriate for studying bounded linear operators. For instance, any linear transformations represented by a matrix that maps bounded sequences to sequences converging to zero can be viewed as an operator from ℓ^∞ to c_0 ([11]) ; ([6]).

Definition 2.2. *A paranorm z on a linear space X, is a function $z : X \to \mathbb{R}$ such that the following conditions are satisfied:*

i. $z(\theta) = 0$*, where θ represents the zero vector;*
ii. $z(x) \geq 0 \quad \forall x \in X$*;*
iii. $z(x) = z(-x)$*;*
iv. $z(x + y) \leq z(x) + z(y)$*;*
v. *If $(\lambda_n)_0^\infty$ is a sequence of scalars with $\lambda_n \to \lambda$ and $(x_n)_0^\infty$ is a sequence in X with $x_n \to x$, then $z(\lambda_n x_n - \lambda_x) \to 0$, thus continuity of multiplication.*

In the context of c_0 and c, which are important to this paper, paranorms can be used to define topologies where standard norms may not be applicable or may be too restrictive. For instance, certain matrix categories of c_0 can form paranormed spaces under functionals induced by summability methods or matrix transformations ([11]); ([8]). The paranorm structure enables the exploration of linear operators, such as the Norlund matrix, under broader convergence condition, specifically in the study of infinite matrices and their spectrum.

Definition 2.3. *A seminorm z on a linear space X, is a function $p : X \to \mathbb{R}$ defined by*

i. $z(x) \geq 0$*;*
ii. $z(x + y) \leq z(x) + z(y)$*;*
iii. $z(\lambda x) = |\lambda| z(x)$*. where* $\lambda \in \mathbb{K}(\mathbb{R}/\mathbb{C})$.

In contrast to a norm, a seminorm has the ability to assign a value of zero to non-zero vectors. This makes seminorms useful for defining topologies where convergence behaviours or equivalence classes are more important than exact distances

([4]); ([10]). Seminorms are the building blocks of locally convex topologies, which are widely used in sequence space theory, distribution theory and dual analysis ([7]); ([11]). In the context of this thesis, seminorms offer a versatile tool for characterizing topologies weaker than the norm topology. The topology of pointwise convergence on these spaces is induced by the family of seminorms:

$$p_k(x) = |x_k|, \quad \text{for each } k \in \mathbb{N}. \tag{2.14}$$

Definition 2.4. *Linear topological space. It is a vector space X over the field $\mathbb{C}$ or $\mathbb{R}$, equipped with a topology τ such that the following conditions are fulfilled:*

a. *The scalar multiplication map*

$$\cdot : \mathbb{K} \times X \to X, \quad (\alpha, y) \mapsto \alpha x, \quad \textit{is continuous}. \tag{2.15}$$

b. *The addition map*

$$+ : X \times X \to X, \quad \textit{defined by} \quad (x, y) \mapsto x + y, \quad \textit{is continuous}. \tag{2.16}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

These conditions ensure that the vector space operations are compatible with the topology, making X a topological vector according to ([9]) and ([11]).

If the topology τ is generated by a norm, then the linear topological space results to a normed space. Also, if the space is complete, then it is a Banach space. Thus, every Banach space is a linear topological space, but not every linear topological space is complete or normable. In particular, c_0 and c are classical examples of linear topological space when equipped with the supremum norm

$$\|x\|_\infty = sup_{n\in\mathbb{N}}|x_n|, \tag{2.17}$$

these spaces become Banach spaces and as well as linear spaces ([11]); ([6]). This topological structure is important when investigating bounded linear operators, such as the newly defined Norlund matrix which acts on these spaces.

2.6. **Spectrum of a Bounded Linear Operator.** Let X be a Banach space over the field $\mathbb{C}$, and let $Z : X \to X$ be a bounded linear operator. The *spectrum* of Z, denoted by $\sigma(Z)$, is the set of all complex numbers $\lambda \in \mathbb{C}$ such that the operator $Z - \lambda I$ is not invertible as an operator on X. That is,

$$\sigma(Z) = \{\lambda \in \mathbb{C} : Z - \lambda I \text{ is not invertible on } X\}. \tag{2.18}$$

By the spectral theory of bounded linear operators on Banach spaces, the spectrum $\sigma(Z)$ is a non-empty, compact subset of the complex plane and satisfies the inclusion:

$$\sigma(Z) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|Z\|\}. \tag{2.19}$$

The spectrum is typically classified into three (3):

a. Point Spectrum ($\sigma_p(Z)$): The set of eigenvalues of Z, thus, all $\lambda \in \mathbb{C}$ such that $Z - \lambda I$ is not injective ([9]).
b. Continuous Spectrum ($\sigma_c(Z)$): The set of $\lambda \in \mathbb{C}$ such that $Z - \lambda I$ is injective and has dense range, but is not subjective ([3]).
c. Residual Spectrum ($\sigma_r(Z)$): The set of $\lambda \in \mathbb{C}$ such that $Z - \lambda I$ is injective, but its range is not dense in X ([11]).

Definition 2.5 (Spectral Radius)**.** *The spectral radius of Z is defined as:*

$$r(Z) = \sup\{|\lambda| : \lambda \in \sigma(Z)\}. \tag{2.20}$$

([3, 9]).

Definition 2.6 (Spectrum of Lower Triangular Matrix)**.** *Let $Z = (z_{nk})$ be a lower triangular matrix operator (i.e., $z_{nk} = 0$ for $k > n$) acting on a Banach space, and assume the rows are uniformly bounded. Then the spectrum of Z is given by:*

$$\sigma(Z) = \overline{\{z_{nn} : n \in \mathbb{N}\}}, \tag{2.21}$$

where z_{nn} denotes the diagonal entries of Z, and the closure is taken in the complex plane ([2, 9, 11]).

Remark 2.1. *For a bounded lower-triangular matrix acting on classical sequence spaces such as c_0 and c, it is known that the spectrum is often equal to the closure of the set of diagonal entries. This principle is applied in Section 3.3 to characterize the full spectrum of the Norlund matrix operator under consideration.*

3. Main results

This section introduces the main results of the paper. We explore the boundedness of the newly defined Norlund Matrix Z as an operator on c_0 and c, determine its eigenvalues and describe its spectrum. The analysis emphasizes the differences in the spectral behavious Z on c_0 and c and supported by a graphical representation of spectrum matrix Z on c_0 and c.

3.1. Boundedness of the Operator Z on c_0 and c.

Lemma 3.1. *Let $Z \in B(c_0)$ and suppose every row sum of Z equals 1. Then for any $x \in c_0$, the sequence Zx converges to 0.*

Proof. Since $x \in c_0$, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for all $n \geq N$. Given that Z has row sums equal to 1, we have:

$$\|Zx\|\infty \leq \|x\|\infty.$$

This implies that $Zx \in c_0$, and therefore, Zx converges to 0. □

Theorem 3.1. *The Norlund matrix operator $Z = (z_{nk})$ is a bounded linear operator on the Banach spaces c_0 and c. That is, $Z \in B(c_0) \cap B(c)$ and $\|Z\| \leq 1$.*

Proof. From equation (2.3) each row of Z has at most two nonzero entries, and the row sums satisfy:

$$\sum_{k=0}^{\infty} |z_{nk}| = \begin{cases} 1 & \text{for n} \geq 0. \end{cases}$$

Thus,

$$\sup_{n\in\mathbf{N}} \sum_{k=0}^{\infty} |Z_{nk}| = 1 < \infty. \tag{3.1}$$

Therefore for all $x \in c_0$ or c, and define the operator $Z = (z_{nk})$ acting on x as:

$$(Zx)_n = \sum_{k=0}^{\infty} z_{nk} x_k. \tag{3.2}$$

Our focus is bounding $|(Zx)_n|$ uniformly over all n. Applying Triangle Inequality, we have

$$|(Zx)_n| = |\sum_{k=0}^{\infty} z_{nk} x_k| \leq \sum_{k=0}^{\infty} |z_{nk}||x_k|. \tag{3.3}$$

Since $x \in c_0$ or c, it is bounded and we define:

$$\|x\|_\infty = \sup_{k\in\mathbf{N}} |x_k|. \tag{3.4}$$

Hence,

$$|x_k| \leq \|x\|_\infty \quad \text{for all k.} \tag{3.5}$$

Putting equation (3.5) into equation (3.3) gives:

$$\sum_{k=0}^{\infty}|z_{nk}||x_k| \leq \|x\|_\infty \sum_{k=0}^{\infty}|z_{nk}|. \tag{3.6}$$

By the boundedness of the row sums,

$$\sum_{k=0}^{\infty}|z_{nk}| = 1 \quad \text{for all n.} \tag{3.7}$$

Hence,

$$|(Zx)_n| \leq \|x\|_\infty \tag{3.8}$$

Taking the supremum over all n, we get

$$\|Zx\|_\infty = \sup_{n\in\mathbb{N}} |(Zx)_n| \leq \|x\|_\infty. \tag{3.9}$$

This proves that Z is a bounded operator on both c_0 and c, with operator norm at most 1:

$$\|Z\| \leq 1. \tag{3.10}$$

Hence, $Z \in B(c_0) \cap B(c)$, which validates the spectral analysis of Z in the framework of bounded linear operators. □

Corollary 3.1. *Under the assumptions of Theorem 3.1, if the Norlund matrix Z has row sums equal to one, then:*

i. *For every $x \in c_0$, we have $Zx \in c_0$, so $Z(c_0) \subseteq c_0$.*
ii. *For every $x \in c$, we have $Zx \in c$, so $Z(c) \subseteq c$.*

3.2. The eigenvalues on c_0 and c.

Theorem 3.2. *Let $p = (p_n)_{n\geq 0}$ be a sequence of non-negative real numbers such that:*

i. $p_0 > 0$,
ii. $P_n = \sum_{k=0}^{n} p_k \to \infty$ *as* $n \to \infty$,
iii. $\lim\limits_{n\to\infty} \dfrac{p_n}{P_n} = 0.$

Define the Norlund matrix $Z = (z_{nk})$ by

$$z_{nk} = \begin{cases} 1 & \text{if } n = 0,\ k = 0, \\ \frac{1}{2} & \text{if } n = 1,\ k = 0, 1, \\ \frac{1}{3} & \text{if } n \geq 2,\ k = n-2,\ n-1,\ n, \\ 0 & \text{otherwise}. \end{cases} \tag{3.11}$$

Then the eigenvalue equation $Zx = \lambda x$ has the following properties:

1. *The matrix Z defines a bounded linear operator on both c_0 and c.*
2. *The matrix Z has no eigenvalues on the sequence space c_0.*
3. *The operator Z has a unique eigenvalue $\lambda = 1$ on the sequence space c, with the corresponding eigenvector $x = (1, 1, 1, \dots)$.*

Proof. We aim to solve the equation $Zx = \lambda x$. Let us apply the matrix form of Z to a general sequence $x = (x_0, x_1, x_2, \dots)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix}. \tag{3.12}$$

The operator Z acts as:

$$(Zx)_n = \begin{cases} x_0 & \text{if n=0}, \\ \frac{1}{2}(x_0 + x_1) & \text{if n=1}, \\ \frac{1}{3}(x_{n-2} + x_{n-1} + x_n) & \text{if n} \geq 2 \end{cases} \tag{3.13}$$

Case 1: $n = 0$

$$(Zx)_0 = x_0 = \lambda x_0 \implies (\lambda - 1)x_0 = 0.$$

So, either $x_0 = 0$ or $\lambda = 1$

Case 2: $n = 1$

$$(Zx)_1 = \frac{1}{2}(x_0 + x_1) = \lambda x_1$$

$$x_0 + x_1 = 2\lambda x_1 \implies x_0 = x_1(2\lambda - 1) \tag{3.14}$$

Case 3: $n = 2$

$$(Zx)_2 = \frac{1}{3}(x_0 + x_1 + x_2) = \lambda x_2$$

(3.15)
$$x_0 + x_1 + x_2 = 3\lambda x_2$$

Putting x_0 from equation (3.14) into (3.15)

$$x_1(2\lambda - 1) + x_1 + x_2 = 3\lambda x_2$$

$$2\lambda x_1 - x_1 + x_1 + x_2 = 3\lambda x_2$$

$$2\lambda x_1 + x_2 = 3\lambda x_2$$

Isolating x_1

$$(2\lambda)x_1 = 3\lambda x_2 - x_2$$

(3.16)
$$x_1 = \frac{(3\lambda - 1)}{2\lambda} x_2$$

Case 4: $n = 3$

$$(Zx)_3 = \frac{1}{3}(x_1 + x_2 + x_3) = \lambda x_3$$

(3.17)
$$x_1 + x_2 + x_3 = 3\lambda x_3$$

Putting equation (3.16) into equation (3.17)

$$\frac{(3\lambda - 1)}{2\lambda} x_2 + x_2 + x_3 = 3\lambda x_3$$

$$\left(\frac{3\lambda - 1}{2\lambda} x_2 + x_2\right) + x_3 = 3\lambda x_3$$

$$x_2 \left(\frac{3\lambda - 1}{2\lambda} + 1\right) + x_3 = 3\lambda x_3$$

$$x_2 \left(\frac{3\lambda - 1 + 2\lambda}{2\lambda}\right) + x_3 = 3\lambda x_3$$

$$x_2 \left(\frac{5\lambda - 1}{2\lambda}\right) = 3\lambda x_3 - x_3$$

$$x_2\left(\frac{5\lambda-1}{2\lambda}\right)=x_3(3\lambda-1). \tag{3.18}$$

If $\lambda\neq 1$, then equation (3.14) results in $x_0=(2\lambda-1)x_1$ and equation (3.15) and (3.17) become difficult recursions that, unless $\lambda=1$ which generally result in non-constant sequences that are bounded or do not converge to zero. Hence, no other values than 1 yields a convergent or null result in c_0 and c unless $x=0$. On the other hand if $\lambda=1$ is a solution then equation (3.14) yields to $x_0=(2(1)-1)x_1=x_1$ and equation (3.15) also yields $x_0+x_1+x_2=3x_2$. Since $x_0=x_1$, let's suppose $x_0=x_1=x_2=c$, then:

$$c+c+c=3c \implies \text{holds}.$$

Same with equation (3.17)

$$x_1+x_2+x_3=3x_3 \implies \text{if} \quad x_1=x_2=x_3=c, \text{then} \quad 3c=3c.$$

All equations are satisfied for $x=(1,1,1,1,\dots)$ where $\lambda=1$.

Hence, on c_0 any non-zero eigenvector must be constant, which is not allowed in c_0, so, no eigenvalues exist on c_0 therefore $Z\in B(c_0)$ possess no eigenvalues. On c a constant sequence $x=(1,1,1,\cdots)$ satisfies $Zx=x$, so $\lambda=1$ is the only eigenvalue on c. □

Remark 3.1. *This theorem demonstrate that under standard conditions of the weight sequence, the Norlund matrix Z exhibits distinct spectral behaviour on the sequence space c_0 and c. The constant sequence $1\in c$ remains uniform, yielding the eigenvalue of 1, whiles c_0 has no eigenvalue.*

Example 1. *Let $p_n=1$ for all $n\geq 0$. Then $P_n=n+1$, and $\frac{p_n}{P_n}=\frac{1}{n+1}\to 0$. The resulting matrix Z is the Cesaro matrix C_1, which satisfies the conditions of the theorem: it has no eigenvalues on c_0 and has eigenvalue of 1 on c.*

3.3. The Spectrum of Norlund Type Matrix Operator $Z=(z_{nk})$ on the Sequence spaces c_0 and c.

3.3.1. *The spectrum of the matrix Z on c_0.*

In this subsection, we investigate the spectrum of the Norlund matrix $Z=(z_{nk})$ as an operator on the Banach space c_0.

Theorem 3.3 (Point Spectrum on c_0)**.** *The point spectrum of the Norlund matrix operator Z on c_0 is empty, and its full spectrum is given by*

$$\sigma(Z|_{c_0}) = [0,1], \quad \sigma_p(Z|_{c_0}) = \emptyset.$$

Proof. Suppose there exists a non-zero vector $x \in c_0$ such that $Zx = \lambda x$ for some $\lambda \in \mathbb{C}$. As previously shown in the proof of Theorem 3.2, the recurrence relation that arises from the eigenvalue equation forces x to be constant sequence:

$$x = (c, c, c, \dots). \tag{3.19}$$

However, constant sequence do not belong to c_0 unless $c = 0$. Hence, Z has no non-zero eigenvectors in c_0 and the point spectrum is empty with the interval $[0,1]$:

$$\sigma(Z|_{c_0}) = [0,1], \quad \sigma_p(Z|_{c_0} = \emptyset). \tag{3.20}$$

□

Remark 3.2. *The absence of eigenvalues for the Norlund matrix Z on c_0 emphasizes on how the structure of the space affects spectral properties. Although constant sequences solve the eigenvalue equation $Zx = \lambda x$, they do not belong to c_0 unless they are trivial.*

Theorem 3.4 (Spectral radius theorem)**.** *Let X be a Banach space and let $Z \in B(X)$ be a bounded linear operator. The spectral radius of Z, $r(Z) \;:=\; \sup\{\, |\lambda| : \lambda \in \sigma(Z)\,\}$, satisfies:*

$$r(Z) \;=\; \lim_{n\to\infty} \|Z^n\|^{1/n} \;=\; \inf_{n\geq 1} \|Z^n\|^{1/n}.$$

In particular, $r(Z) \leq \|Z\|$, and $\sigma(Z)$ is a nonempty compact subset of the closed disk $\{\, \lambda \in \mathbb{C} : \, |\lambda| \leq \|Z\|\,\}$.

Proof. Since Z is a bounded linear operator on the sequence space c_0 with $\|Z\| \leq 1$, by Theorem 3.4, the spectral radius satisfies:

$$r(Z) = \lim_{n\to\infty} \|Z^n\|^{1/n} \leq 1, \tag{3.21}$$

and thus the spectrum is contained within the closed unit disk:

$$\sigma(Z|_{c_0}) \subseteq \{\lambda \in \mathbb{C} : |\lambda \leq 1\}. \tag{3.22}$$

□

Remark 3.3. *The spectral radius result provides an essential bound for the spectrum of the operator Z on c_0. Knowing that $\|Z\| \leq 1$ immediately restricts the spectrum to lie within the closed unit disk in the complex plane ([3, 9]). This confirms that no spectral value can have modulus greater than 1, which aligns with the observed diagonal entries and their accumulation behavior. Such bounds are valuable in operator theory as they offer insight into stability and long-term behavior of iterated operators.*

Theorem 3.5 (Full spectrum via diagonal entries)**.** *Let $Z = (z_{nk})$ be a bounded lower triangular matrix operator on a Banach sequence space c_0. That is, $z_{nk} = 0$ whenever $k > n$, and the rows of Z are uniformly bounded. Then the spectrum of Z is given by the closure of its diagonal entries: $\sigma(Z) = \overline{\{ z_{nn} : n \in \mathbb{N} \}}$.*

Proof. Applying the result for lower triangular matrices from Definition 2.6, we consider the diagonal entries of Z, which are given by:

$$\operatorname{diag}(Z) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots \right). \tag{3.23}$$

This sequence accumulates at 0 with the interval of $[0, 1]$. Hence, the full spectrum of Z on c_0 is;

$$\sigma(Z|_{c_0}) = \overline{\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}} = [0, 1]. \tag{3.24}$$

□

Remark 3.4. *This result enables the determination of the spectrum of certain structured operators, such as the Norlund matrix Z, by examining the limit points of its diagonal entries and it has been established that $\|Z\| \leq 1$. Hence, by Definition 2.6, the spectrum of Z is contained within the closed unit disk:*

$$\sigma(Z|_{c_0}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}. \tag{3.25}$$

This provides a bound for the location of the spectrum.

3.3.2. *The spectrum of the matrix Z on c.*

Proposition 3.1. *The spectrum of the operator Z on the space c is the interval $[0, 1]$, and its point spectrum consists of the singleton $\{1\}$:*

$$\sigma(Z|_c) = [0, 1], \quad \sigma_p(Z|_c) = \{1\}.$$

Proof. As before, $Z \in B(c)$ and $\|Z\| \leq 1$, so

$$\sigma(Z|_c) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

The constant sequence $(x = 1, 1, 1, \dots) \in c$ satisfies

$$Zx = x \implies \lambda = 1.$$

Hence, $\lambda = 1$ is an eigenvalue with corresponding eigenvector in c, $\sigma_p(Z|_c) = \{1\}$.

To describe the full spectrum, we again apply Theorem 3.5 to the structure Z. Since the diagonal entries $z_{nn} \to 0$ as $n \to \infty$, and Z is lower-triangular with bounded row sums, the spectrum on c must include the closure of these diagonal values. Hence, the closure of this set gives:

$$\sigma(Z|_c) = \overline{\{z_{nn} : n \in \mathbb{N}\}} = [0, 1].$$

□

Graphical Representation of spectrum matrix Z on c_0 and c

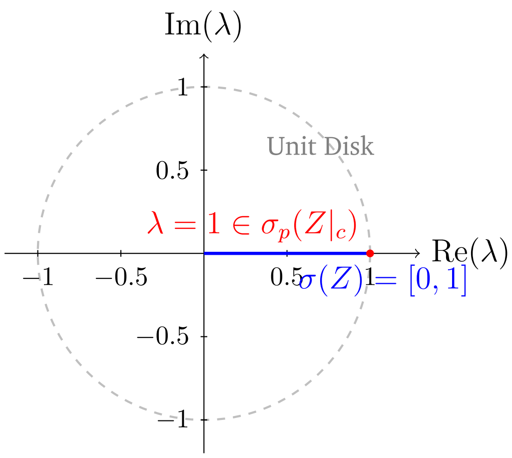

FIGURE 1. Spectrum of the Norlund matrix Z on the complex plane. As shown, the spectrum the newly defined Norlund matrix Z lies entirely within the interval $[0, 1]$ on the real axis. This reflects the fact that the newly defined Norlund matrix is a compact operator with spectral values accumulating at 0. The red point at $\lambda = 1$ indicated the presence of an eigenvalue in the point spectrum $\sigma_p(Z|c)$, which corresponds to a nontrivial solution of the equation $Zx = x$ in the space c. No such eigenvalue exists in c_0, highlighting the topological distinctions between the two spaces.

CONFLICT OF INTEREST

The authors declare no competing interests.

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