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SUBCLASS OF ANALYTIC FUNCTIONS RELATED TO MITTAG-LEFFLER FUNCTION

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ABSTRACT. The subclass of analytic functions related to the Mittag-Leffler function can be explored through the lens of fractional calculus and complex analysis. The Mittag-Leffler function generalizes the exponential function and plays a crucial role in various fields such as fractional differential equations and complex analysis.The target of this paper is to introduce a new subclass and obtained coefficient bounds, distrotion properties, radii of starlike, convex and close-to-convex, extreme points, hadamard product and closure theorems.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0) = 0$ and $u'(0) = 1$. We denote by S the subclass of $\mathcal{A}$ consisting of $u(z)$ which are all

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univalent in E. A function $u \in \mathcal{A}$ is a starlike function of the order $\upsilon(0 \leq \upsilon < 1)$ if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon,\ (z \in E). \tag{1.2}$$

We denote it by the class $S^*(\upsilon)$. A function $u \in \mathcal{A}$ is a convex function of the order $\upsilon(0 \leq \upsilon < 1)$ if it satisfies

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \upsilon,\ (z \in E). \tag{1.3}$$

We denote this class with $K(\upsilon)$.

For $u \in \mathcal{A}$ given by (1.1)and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.4}$$

Their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z),\ (z \in E). \tag{1.5}$$

Note that $u * g \in \mathcal{A}$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ (a_n \geq 0,\ z \in E) \tag{1.6}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon)$, $C(\upsilon) = T \cap K(\upsilon)$.

The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [16] and Orhan [14].

The study of operators is fundamental in geometric function theory, complex analysis, and related areas. Several derivative and integral operators can be expressed by convolution of certain analytic functions. It should be noted that this formalism helps future mathematical research as well as a better grasp of the geometric properties of such operators. The following defines the familiar Mittag-Leffler function $E_\upsilon(z)$ introduced by Mittag- Leffler [12] and its generalization $E_{\upsilon,\tau}(z)$ introduced by Wiman [19],
$E_\upsilon(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\upsilon n+1)}$ and $E_{\upsilon,\tau}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\upsilon n+\tau)}$, where $\upsilon, \tau \in \mathbb{C}, \Re(\upsilon) > 0$ and

$\Re(\tau) > 0$. We define the function $Q_{\upsilon,\tau}(z)$ by

$$Q_{\upsilon,\tau}(z) = z\Gamma(\tau)E_{\upsilon,\tau}(z).$$

Observe that the function $E_{\upsilon,\tau}$ contains many well-known functions as its special case, for example,

$$E_{1,1}(z) = \mathrm{e}^z,\ E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z},$$
$$E_{2,1}\left(z^2\right) = \cosh z,\ E_{2,1}\left(-z^2\right) = \cos z,\ E_{2,2}\left(z^2\right) = \frac{\sinh z}{z},$$
$$E_{2,2}\left(-z^2\right) = \frac{\sin z}{z}c,\ E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right]$$
$$\text{and } E_4(z) = \frac{1}{2}\left[\cos z^{1/4} + \cosh z^{1/4}\right].$$

The Mittag-Leffler function appears naturally in the solution of fractional order differential and integral equations. In the study of complex systems and super diffusive transport, in particular, fractional generalisation of the kinetic equation, random walks, and Levy flights. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found, e.g., in [1, 3–9].

Now, for $f \in A$, we define the following differential operator $\mathfrak{D}_\lambda^m(\upsilon,\tau)u : A \to A$ by

$$\begin{aligned}
\mathfrak{D}_\lambda^0(\upsilon,\tau)u(z) &= u(z) * Q_{\upsilon,\tau}(z),\\
\mathfrak{D}_\lambda^1(\upsilon,\tau)u(z) &= (1-\lambda)(u(z) * Q_{\upsilon,\tau}(z)) + \lambda z(u(z) * Q_{\upsilon,\tau}(z))',\\
&\ \vdots\\
\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z) &= \mathfrak{D}_\lambda^1\left(\mathfrak{D}_\lambda^{m-1}(\upsilon,\tau) f(z)\right).
\end{aligned}$$

If u is given by (1.1) then from the definition of the operator $\mathfrak{D}_\lambda^m u$ it is easy to see that

$$\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z) = z + \sum_{n=2}^{\infty} \phi_n^m(\lambda,\upsilon,\tau)a_n z^n, \tag{1.7}$$

where

$$\phi_n^m(\lambda,\upsilon,\tau) = \frac{\Gamma(\tau)}{\Gamma(\upsilon(n-1)+\tau)}[\lambda(n-1)+1]^m. \tag{1.8}$$

Note that

(1) when $\upsilon = 0$ and $\tau = 1$, we get Al-Oboudi operator [2],

(2) when $\upsilon = 0, \tau = 1$ and $\lambda = 1$, we get Salagean operator [17],

(3) when $m = 0$, we get $E_{\upsilon,\tau}(z)$, Srivastava et al. [18].

Now, by making use of the differential operator $\mathfrak{D}_\lambda^m(\upsilon,\tau)u$, we define a new subclass of functions belonging to the class A.

Definition 1.1. *For* $0 \leq \hbar < 1$, $0 \leq \sigma < 1$, *and* $0 < \varsigma < 1$, *we denote by* $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ *the subclass of* u *consisting of functions of the form* (1.6), *whose geometric condition satisfies*

$$\left| \frac{\hbar \left((\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))' - \dfrac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z} \right)}{\sigma(\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))' + (1-\hbar)\dfrac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z}} \right| < \varsigma, \quad z \in E,$$

where $\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)$ *is given by* (1.7).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$.

Theorem 2.1. *Let the function* u *be defined by* (1.6). *Then* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ *if and only if*

$$\sum_{n=2}^{\infty} [\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)a_n \leq \varsigma(\sigma + (1-\hbar)), \tag{2.1}$$

where $0 < \varsigma < 1, 0 \leq \hbar < 1$ *and* $0 \leq \sigma < 1$. *The result* (2.1) *is sharp for the function*

$$u(z) = z - \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)} z^n, \ n \geq 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned} &\left| \hbar \left((\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))' - \frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z} \right) \right| \\ &- \varsigma \left| \sigma \left(\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))' + (1-\hbar)\frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z} \right) \right| \\ &= \left| -\hbar \sum_{n=2}^{\infty} (n-1)\phi_n^m(\lambda,\upsilon,\tau)a_n z^{n-1} \right| \end{aligned}$$

$$
\begin{aligned}
&-\varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\phi_n^m(\lambda,\upsilon,\tau)a_nz^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)a_n-\varsigma(\sigma+(1-\hbar)) \\
&\leq\ 0.
\end{aligned}
$$

Hence, by maximum modulus principle, $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. Now assume that $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ so that

$$
\left|\frac{\hbar\left((\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))'-\frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z}\right)}{\sigma(\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))'+(1-\hbar)\frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z}}\right|<\varsigma,\ z\in E.
$$

Hence

$$
\begin{aligned}
&\left|\hbar\left((\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))'-\frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z}\right)\right| \\
&<\varsigma\left|\sigma\left(\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z))'+(1-\hbar)\frac{\mathfrak{D}_\lambda^m(\upsilon,\tau)u(z)}{z}\right)\right|.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\hbar(n-1)\phi_n^m(\lambda,\upsilon,\tau)a_nz^{n-1}\right| \\
&<\varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\phi_n^m(\lambda,\upsilon,\tau)a_nz^{n-1}\right|.
\end{aligned}
$$

Thus

$$
\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)a_n\leq\varsigma(\sigma+(1-\hbar))
$$

and this completes the proof. □

Corollary 2.1. *Let the function* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then*

$$
a_n\leq\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}z^n,\ n\geq 2.
$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$.

Theorem 3.1. *Let the function* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then*

$$|z| - \frac{\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2 \leq |u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.$$

The result is sharp and attained for

$$u(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}z^2.$$

Proof.

$$|u(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n\right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}. \tag{3.1}$$

Thus

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.$$

Also

$$|u(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.$$

□

Theorem 3.2. *Let* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then*

$$1 - \frac{2\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z| \leq |u'(z)| \leq 1 + \frac{2\varsigma(\sigma+(1-\hbar))}{\phi_2^m(\lambda,\upsilon,\tau)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|,$$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma + (1-\hbar))}{\phi_2^m(\lambda, \upsilon, \tau)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} z^2.$$

Proof. Notice that

$$\begin{aligned} &\phi_2^m(\lambda, \upsilon, \tau)[\hbar + \varsigma(2\sigma + 1 - \hbar)] \sum_{n=2}^{\infty} n a_n \\ &\leq \sum_{n=2}^{\infty} n[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda, \upsilon, \tau) a_n \\ &\leq \varsigma(\sigma + (1-\hbar)), \end{aligned} \tag{3.2}$$

from Theorem 2.1. Thus

$$\begin{aligned} |u'(z)| &= \left|1 - \sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n a_n \leq 1 + |z| \frac{2\varsigma(\sigma + (1-\hbar))}{\phi_2^m(\lambda, \upsilon, \tau)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}. \end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned} |u'(z)| &= \left|1 - \sum_{n=2}^{\infty} n a_n z^{n-1}\right| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\geq 1 - |z| \sum_{n=2}^{\infty} n a_n \geq 1 - |z| \frac{2\varsigma(\sigma + (1-\hbar))}{\phi_2^m(\lambda, \upsilon, \tau)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar, \sigma, \varsigma)$.

Theorem 4.1. *Let $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar, \sigma, \varsigma)$. Then u is starlike in $|z| < R_1$ of order ρ, $0 \leq \rho < 1$, where*

$$R_1 = \inf_n \left\{ \frac{(1-\rho)(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\phi_n^m(\lambda, \upsilon, \tau)}{(n-\rho)\varsigma(\sigma + (1-\hbar))} \right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{4.1}$$

Proof. u is starlike of order $\rho,\ 0 \leq \rho < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)} - 1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leq 1-\rho \; if \; \sum_{n=2}^{\infty}\frac{(n-\rho)}{(1-\rho)}a_n|z|^{n-1} \leq 1. \tag{4.2}$$

Hence by Theorem 2.1, (4.2) will be true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{(n-\rho)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}},\ n \geq 2. \tag{4.3}$$

The theorem follows easily from (4.3). □

Theorem 4.2. *Let* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then* u *is convex in* $|z| < R_2$ *of order* $\rho,\ 0 \leq \rho < 1,$ *where*

$$R_2 = \inf_n \left\{\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{n(n-\rho)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}},\ n \geq 2. \tag{4.4}$$

Proof. u is convex of order $\rho,\ 0 \leq \rho < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1-\rho \; if \; \sum_{n=2}^{\infty}\frac{n(n-\rho)}{(1-\rho)}a_n|z|^{n-1} \leq 1. \tag{4.5}$$

Hence by Theorem 2.1, (4.5) will be true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{n(n-\rho)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

Theorem 4.3. *Let* $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then* u *is close-to-convex in* $|z| < R_3$ *of order* $\rho,\ 0 \leq \rho < 1,$ *where*

$$R_3 = \inf_n \left\{\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{n\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \; n \geq 2. \tag{4.7}$$

Proof. u is close-to-convex of order $\rho, 0 \leq \rho < 1$ if

$$\Re\{u'(z)\} > \rho.$$

Thus it is enough to show that

$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus

$$|u'(z)-1| \leq 1-\rho \; if \; \sum_{n=2}^{\infty}\frac{n}{(1-\rho)}a_n|z|^{n-1} \leq 1. \tag{4.8}$$

Hence by Theorem 2.1, (4.8) will be true if

$$\frac{n}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{n\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{4.9}$$

The theorem follows easily from (4.9). □

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$.

Theorem 5.1. *Let* $u_1(z)=z$ *and*

$$u_n(z)=z-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}z^n,\ \textit{for}\ n=2,3,\cdots.$$

Then $u\in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ *if and only if it can be expressed in the form*

$$u(z)=\sum_{n=1}^{\infty}\theta_n u_n(z),\ where\ \theta_n\geq 0\ and\ \sum_{n=1}^{\infty}\theta_n=1.$$

Proof. Assume that $u(z)=\sum_{n=1}^{\infty}\theta_n u_n(z)$, hence we get

$$u(z)=z-\sum_{n=2}^{\infty}\frac{\varsigma(\sigma+(1-\hbar))\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}z^n.$$

Now, $u\in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$, since

$$\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}\times\frac{\varsigma(\sigma+(1-\hbar))\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}$$
$$=\sum_{n=2}^{\infty}\theta_n=1-\theta_1\leq 1.$$

Conversely, suppose $u\in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty}\theta_n u_n(z)$.

Now $u\in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ implies from Theorem 2.1

$$a_n\leq\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}.$$

Setting $\theta_n=\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}a_n, n=2,3,\cdots$ and $\theta_1=1-\sum_{n=2}^{\infty}\theta_n$, we obtain $u(z)=\sum_{n=1}^{\infty}\theta_n u_n(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$.

Theorem 6.1. *Let* $u, g \in TS(\hbar,\sigma,\varsigma,\vartheta)$. *Then* $u * g \in TS(\hbar,\sigma,\zeta,\vartheta)$ *for*

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } (u*g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\hbar))\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \phi_n^m(\lambda,\upsilon,\tau) - \varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. $u \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma + (1-\hbar))} a_n \leq 1 \tag{6.1}$$

and

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma + (1-\hbar))} b_n \leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\zeta(\sigma + (1-\hbar))} a_n b_n \leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma + (1-\hbar))} \sqrt{a_n b_n} \leq 1. \tag{6.4}$$

Therefore it is enough to show that

$$\begin{aligned} &\frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\zeta(\sigma + (1-\hbar))} a_n b_n \\ \leq &\frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma + (1-\hbar))} \sqrt{a_n b_n}. \end{aligned}$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\zeta}{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\varsigma}. \tag{6.5}$$

From (6.4)

$$\sqrt{a_n b_n} \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}.$$

Thus it is enough to show that

$$\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\hbar))\hbar(n-1)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\phi_n^m(\lambda,\upsilon,\tau)-\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

□

7. Closure Theorems

We shall prove the following closure theorems for the class $TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$.

Theorem 7.1. *Let* $u_j \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma), j=1,2,\ldots,s$. *Then*

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$$

For $u_j(z)=z-\sum_{n=2}^{\infty} a_{n,j}z^n$, *where* $\sum_{j=1}^{s} c_j=1.$

Proof. We have

$$g(z)=\sum_{j=1}^{s} c_j u_j(z)=z-\sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n = z-\sum_{n=2}^{\infty} e_n z^n,$$

where $e_n=\sum_{j=1}^{s} c_j a_{n,j}$. Thus $g(z)\in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))} e_n \leq 1,$$

that is, if

$$\sum_{n=2}^{\infty}\sum_{j=1}^{s} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))} c_j a_{n,j}$$

$$=\sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))} a_{n,j}$$
$$\leq \sum_{j=1}^{s} c_j = 1.$$

□

Theorem 7.2. *Let* $u, g \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$. *Then*

$$h(z) = z - \sum_{n=2}^{\infty}(a_n^2 + b_n^2)z^n \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma),$$

where

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\phi_n^m(\lambda,\upsilon,\tau) - 2\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

Proof. Since $u, g \in TS_{\lambda,\upsilon}^{m,\tau}(\hbar,\sigma,\varsigma)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}\right]^2(a_n^2+b_n^2) \leq 1. \tag{7.1}$$

But $h(z) \in TS(\hbar,\sigma,\zeta,q,m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\zeta(\sigma+(1-\hbar))}(a_n^2+b_n^2) \leq 1, \tag{7.2}$$

where $0<\zeta<1$, however (7.1) implies (7.2) if

$$\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\phi_n^m(\lambda,\upsilon,\tau)}{\zeta(\sigma+(1-\hbar))}$$
$$\leq \frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\phi_n^m(\lambda,\upsilon,\tau)}{\varsigma(\sigma+(1-\hbar))}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\phi_n^m(\lambda,\upsilon,\tau)-2\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

□

Conclusion: In this paper, we have introduced and studied a specific subclass of analytic functions defined through the use of a Mittag-Leffler function. By leveraging the properties of this Mittag-Leffler function, we have been able to derive several interesting results regarding the coefficient estimates, growth and distortion theorems, and radii of starlikeness and convexity for the functions within this subclass. The study of this specific subclass of analytic functions defined by Mittag-Leffler function not only enhances our theoretical understanding but also has significant implications for applied mathematics and other scientific domains. Further research in this area promises to yield even richer and more comprehensive insights.

Conflict of interest

The authors declare no conflict of interest.

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