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ON SOME BOUNDS INVOLVING THE CONVOLUTION PRODUCT UNDER THE CONVEXITY ASSUMPTION

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ABSTRACT. Convex functions and the convolution product are two key mathematical concepts that are rarely studied together. In this article, we explore the relationship between them and establish several comprehensive lower and upper bounds. Complete and detailed proofs are provided for all the main results.

1. INTRODUCTION

Convex functions play a fundamental role in mathematics, particularly in optimization and analysis. A standard definition is given below, with the entire real line $\mathbb{R}$ taken as the domain of definition.

Definition 1.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if, for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

Intuitively, a function is convex if the line segment joining any two points on its graph lies above the graph itself. Basic examples of convex functions on $\mathbb{R}$ include $f(x) = x^2$, $f(x) = e^x$ and $f(x) = e^{-x}$. As a notable property, if a function f is twice

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differentiable on $\mathbb{R}$, then f is convex on $\mathbb{R}$ if and only if, for any $x \in \mathbb{R}$, $f''(x) \geq 0$. A function is said to be concave if the inequality in Equation (1.1) is reversed.

Convex functions are closely connected to many classical inequalities. Among the most famous are the Jensen integral inequality and the Hermite-Hadamard integral inequality. Both have inspired numerous refinements, extensions, and generalizations over the years. Comprehensive treatments and further developments on convex functions and their related inequalities can be found in the literature. See, e.g., [1–4, 6–15].

While convexity provides powerful tools for deriving inequalities, the convolution product of two functions is another key concept that plays a central role in analysis, probability, and signal processing. A standard definition is given below, with the entire real line $\mathbb{R}$ taken as the domain of definition.

Definition 1.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Then the convolution product of f and g is defined by the function $f \star g : \mathbb{R} \to \mathbb{R}$ such that, for any $x \in \mathbb{R}$,*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt,$$

provided that it is well-defined.

As a basic property, the convolution product is commutative, i.e., $f \star g = g \star f$, and associative when all integrals exist. The convolution product is particularly essential in Fourier analysis, as the Fourier transform of a convolution product is the product of the Fourier transforms of the individual functions. See, e.g., [5, 16].

In this article, we investigate some inequalities involving the convolution product under the assumption that one of the functions is convex. To the best of our knowledge, this aspect has received little attention in the existing literature, and many of the results presented here appear to be new. Our goal is to establish general inequalities that reveal how the convexity of one function influences the behavior of the convolution product, with potential applications in analysis, probability theory, and related areas.

The remainder of this article is organized as follows: Section 2 presents the main results, including detailed proofs and illustrative examples. Section 3 is devoted to concluding remarks and a discussion of possible directions for future research.

2. Results, proofs and examples

Preliminary note: For the purposes of this article, it is assumed that any integral or convolution product used is well-defined.

The theorem below illustrates an important property of convexity for the convolution product.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ be two functions. Assume that f is convex. Then $f \star g$ is also convex.*

Assuming that f is concave rather than convex results in $f \star g$ also being concave.

Proof. Using the classical convex inequality in Equation (1.1) and the fact that g is non-negative, for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$(f \star g)(\lambda x + (1-\lambda)y) = \int_{-\infty}^{\infty} f(\lambda x + (1-\lambda)y - t)g(t)dt$$
$$= \int_{-\infty}^{\infty} f(\lambda(x-t) + (1-\lambda)(y-t))g(t)dt$$
$$\leq \int_{-\infty}^{\infty} (\lambda f(x-t) + (1-\lambda)f(y-t))\, g(t)dt$$
$$= \lambda \int_{-\infty}^{\infty} f(x-t)g(t)dt + (1-\lambda) \int_{-\infty}^{\infty} f(y-t)g(t)dt$$
$$= \lambda(f \star g)(x) + (1-\lambda)(f \star g)(y).$$

The classical convex inequality is satisfied, proving that $f \star g$ is convex. Assuming that f is concave rather than convex results in the reverse of the final inequality, meaning that $f \star g$ is also concave. This completes the proof. □

Thanks to this result, under the assumption that f is convex and g non-negative, we can apply the Jensen integral inequality and the Hermite-Hadamard integral inequality to the convolution product $f \star g$. In particular, the Hermite-Hadamard integral inequality gives

$$(f \star g)\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b (f \star g)(x)dx \leq \frac{(f \star g)(a) + (f \star g)(b)}{2},$$

where $a, b \in \mathbb{R}$ such that $a < b$.

The theorem below provides a lower bound for the convolution product. Its proof relies primarily on a careful application of the Jensen integral inequality.

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ be two functions. Assume that f is convex, g is integrable and $tg(t)$ is integrable. Then, for any $x \in \mathbb{R}$, we have*

$$(f \star g)(x) \geq f\left(x - \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt}\right) \int_{-\infty}^{\infty} g(t)dt.$$

Assuming that f is concave rather than convex reverses this inequality.

Proof. Since g is non-negative, the function $w : \mathbb{R} \to [0, \infty)$ such that, for any $t \in \mathbb{R}$, $w(t) = g(t)/\int_{-\infty}^{\infty} g(u)du$, defines a probability weight function. Applying the Jensen integral inequality to this weight function and the convex function f, we obtain

$$\begin{aligned}
(f \star g)(x) &= \int_{-\infty}^{\infty} f(x-t)g(t)dt \\
&= \left(\int_{-\infty}^{\infty} g(u)du\right) \int_{-\infty}^{\infty} f(x-t) \frac{g(t)}{\int_{-\infty}^{\infty} g(u)du} dt \\
&= \left(\int_{-\infty}^{\infty} g(u)du\right) \int_{-\infty}^{\infty} f(x-t)w(t)dt \\
&\geq \left(\int_{-\infty}^{\infty} g(u)du\right) f\left(\int_{-\infty}^{\infty} (x-t)w(t)dt\right) \\
&= \left(\int_{-\infty}^{\infty} g(u)du\right) f\left(\int_{-\infty}^{\infty} (x-t) \frac{g(t)}{\int_{-\infty}^{\infty} g(u)du} dt\right) \\
&= \left(\int_{-\infty}^{\infty} g(u)du\right) f\left(x \int_{-\infty}^{\infty} \frac{g(t)}{\int_{-\infty}^{\infty} g(u)du} dt - \int_{-\infty}^{\infty} \frac{tg(t)}{\int_{-\infty}^{\infty} g(u)du} dt\right) \\
&= \left(\int_{-\infty}^{\infty} g(u)du\right) f\left(x - \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(u)du}\right) \\
&= f\left(x - \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt}\right) \int_{-\infty}^{\infty} g(t)dt.
\end{aligned}$$

Assuming that f is concave rather than convex allows us to apply the reverse Jensen integral inequality, which leads to the reverse of the final inequality. This ends the proof. □

It is interesting to note that, when integrated, the upper and lower bounds in Theorem 2.2 coincides, i.e., by the Fubini integral theorem, we have

$$\int_{-\infty}^{\infty}(f\star g)(x)dx = \left(\int_{-\infty}^{\infty} f(x)dx\right)\left(\int_{-\infty}^{\infty} g(x)dx\right)$$
$$= \left(\int_{-\infty}^{\infty} f\left(x - \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt}\right)dx\right)\left(\int_{-\infty}^{\infty} g(t)dt\right).$$

The theorem below establishes a lower bound for the convolution product under the assumption that g is an even function. Two different proofs are provided: one based on Theorem 2.2, and the other using elementary arguments, thereby reinforcing the result.

Theorem 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ be two functions. Assume that f is convex, and g is even and integrable. Then, for any $x \in \mathbb{R}$, we have*

$$(f\star g)(x) \geq 2f(x)\int_{0}^{\infty} g(t)dt.$$

Assuming that f is concave rather than convex reverses this inequality.

Proof. Two different proofs are proposed, the first one assuming an additonal assumption on g.

Proof 1: Using an additional assumption and Theorem 2.2. Under the additional assumption that $tg(t)$ is integrable (that must be considered in the statement), it follows from Theorem 2.2 that

$$(f\star g)(x) \geq f\left(x - \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt}\right)\int_{-\infty}^{\infty} g(t)dt.$$

Since g is integrable, $tg(t)$ is integrable and g is even, we have

$$\int_{-\infty}^{\infty} tg(t)dt = 0, \quad \int_{-\infty}^{\infty} tg(t)dt = 2\int_{0}^{\infty} tg(t)dt.$$

This implies that

$$(f\star g)(x) \geq 2f(x)\int_{0}^{\infty} g(t)dt.$$

Assuming that f is concave rather than convex reverse this inequality, as stated in Theorem 2.2.

Proof 2: Ignoring Theorem 2.2. Applying a suitable decomposition of the integral, making the change of variables $v = -t$, taking into account that g is even, using the classical convex inequality in Equation (1.1) with $\lambda = 1/2$ together with the fact that g is non-negative, and applying a classical integral property for the even function g, we obtain

$$\begin{aligned}(f \star g)(x) &= \int_{-\infty}^{\infty} f(x-t)g(t)dt \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x-t)g(t)dt + \int_{-\infty}^{\infty} f(x-t)g(t)dt\right) \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x-t)g(t)dt + \int_{-\infty}^{\infty} f(x+v)g(-v)dv\right) \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x-t)g(t)dt + \int_{-\infty}^{\infty} f(x+v)g(v)dv\right) \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x-t)g(t)dt + \int_{-\infty}^{\infty} f(x+t)g(t)dt\right) \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(f(x-t) + f(x+t)\right) g(t)dt \\ &\geq \int_{-\infty}^{\infty} f\left(\frac{x-t+x+t}{2}\right) g(t)dt \\ &= f(x)\int_{-\infty}^{\infty} g(t)dt = 2f(x)\int_{0}^{\infty} g(t)dt.\end{aligned}$$

Assuming that f is concave rather than convex allows us to apply the classical concave inequality instead of the classical convex inequality. This results in the reverse of the final inequality. This concludes the proof. □

This theorem demonstrates the significance of the convex function f in establishing a lower bound for the convolution product $f \star g$.

Two concrete examples of Theorem 2.3 are now presented.

Example 1. *We consider $f(x) = x^2$ and $g(x) = e^{-x^2}$, which satisfy the required assumptions of Theorem 2.3. Then we have*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{-\infty}^{\infty} (x-t)^2 e^{-t^2} dt$$

$$= \int_{-\infty}^{\infty} (x^2 - 2xt + t^2) e^{-t^2} dt$$
$$= x^2 \int_{-\infty}^{\infty} e^{-t^2} dt - 2x \int_{-\infty}^{\infty} t e^{-t^2} dt + \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$
$$= x^2\sqrt{\pi} - 2x \times 0 + \frac{\sqrt{\pi}}{2} = \sqrt{\pi}\left(x^2 + \frac{1}{2}\right).$$

On the other hand, the lower bound is

$$2f(x)\int_0^{\infty} g(t)dt = 2x^2 \int_0^{\infty} e^{-t^2} dt = 2x^2 \frac{\sqrt{\pi}}{2} = x^2\sqrt{\pi}.$$

We clearly have

$$(f \star g)(x) = \sqrt{\pi}\left(x^2 + \frac{1}{2}\right) \geq x^2\sqrt{\pi} = 2f(x)\int_0^{\infty} g(t)dt,$$

supporting the result of Theorem 2.3.

Example 2. *We consider $f(x) = x^2$ and $g(x) = e^{-|x|}$, which satisfy the required assumptions of Theorem 2.3. Then we have*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{-\infty}^{\infty} (x-t)^2 e^{-|t|} dt$$
$$= \int_{-\infty}^{\infty} (x^2 - 2xt + t^2) e^{-|t|} dt$$
$$= x^2 \int_{-\infty}^{\infty} e^{-|t|} dt - 2x \int_{-\infty}^{\infty} t e^{-|t|} dt + \int_{-\infty}^{\infty} t^2 e^{-|t|} dt$$
$$= x^2 \times 2 - 2x \times 0 + 4 = 2x^2 + 4.$$

On the other hand, the lower bound is

$$2f(x)\int_0^{\infty} g(t)dt = 2x^2 \int_0^{\infty} e^{-t} dt = 2x^2.$$

We clearly have

$$(f \star g)(x) = 2x^2 + 4 \geq 2x^2 = 2f(x)\int_0^{\infty} g(t)dt,$$

supporting the result of Theorem 2.3.

The theorem below establishes an inequality connecting the convolution product and a convex function in the context of three interacting functions. Its proof relies primarily on a careful application of the Jensen integral inequality.

Theorem 2.4. *Let $f, h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ be three functions. Assume that f is convex, and g is integrable such that $\int_{-\infty}^{\infty} g(t)dt = 1$. Then, for any $x \in \mathbb{R}$, we have*

$$f\left((h \star g)(x)\right) \leq ((f \circ h) \star g)(x),$$

where $f \circ h$ denotes the composition of the functions f and h, in order.

Assuming that f is concave rather than convex reverses this inequality.

Proof. Since g is non-negative with $\int_{-\infty}^{\infty} g(t)dt = 1$, it defines a probability weight function. Applying the Jensen integral inequality to this weight function and the convex function f, we obtain

$$\begin{aligned}
&f\left((h \star g)(x)\right) = f\left(\int_{-\infty}^{\infty} h(x-t)g(t)dt\right) \\
&\leq \int_{-\infty}^{\infty} f\left(h(x-t)\right) g(t)dt = \int_{-\infty}^{\infty} (f \circ h)(x-t)g(t)dt \\
&= ((f \circ h) \star g)(x).
\end{aligned}$$

Assuming that f is concave rather than convex allows us to apply the reverse Jensen integral inequality, which leads to the reverse of the final inequality. This concludes the proof. □

In particular, when $h = f$, we have the elegant inequality, for any $x \in \mathbb{R}$,

$$f\left((f \star g)(x)\right) \leq ((f \circ f) \star g)(x).$$

3. Conclusion

The results presented in this article highlight a connection between convex functions and the convolution product, leading to several new integral inequalities. These findings suggest that convex analysis can offer deeper insights into convolution-type operations, particularly in functional and harmonic analysis. Future research could explore extensions to higher-dimensional settings, weighted convolutions, or inequalities involving generalized convexities such as s-convex or log-convex functions. Another promising direction is to investigate probabilistic interpretations and applications of these inequalities in information theory and stochastic processes.

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