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ASYMPTOTIC ANALYSIS OF SUSTAINABLE EDUCATIONAL QUALITY: STABILITY AND CONVERGENCE IN A NONLINEAR DELAY MODEL

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ABSTRACT. This study introduces a third-order nonlinear delay differential equation (DDE) model designed to simulate the dynamics of educational quality, aligning specifically with the goals of Sustainable Development Goal 4 (SDG 4). The model is structured to incorporate time lags which accurately reflect real-world systemic delays in areas such as policy implementation, resource allocation, and feedback from learning outcomes. Educational quality is the primary dependent variable, with the DDE's delay components capturing the inertia inherent in complex educational systems. Using the rigorous Lyapunov's direct method, augmented with integral terms, the research theoretically proves the uniform asymptotic stability and convergence of solution pairs under bounded conditions. This theoretical finding is crucial, as it ensures that initial disparities or differences between various educational states will asymptotically diminish over time, converging toward a stable, common equilibrium. The study validates these theoretical results through extensive numerical simulations performed in Mathematica. These simulations, which include phase portraits and qualitative analysis across varying delay magnitudes, confirm that small time delays are critical for promoting stable, bounded solutions and achieving rapid convergence to the desired quality standard.

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1. Introduction

The **Sustainable Development Goals (SDGs)**, established by the United Nations in 2015, provide a global blueprint for addressing critical challenges like poverty and inequality [1], [2]. **SDG 4**, which focuses on ensuring inclusive and equitable quality education, is considered a foundational pillar for achieving overall sustainability [3]. **Mathematical modeling** [4] is an essential tool for advancing SDG objectives in education by simulating complex scenarios, optimizing resource allocation, and forecasting long-term outcomes. Recent efforts emphasize integrating mathematical approaches into sustainability education to foster critical thinking and problem-solving skills necessary for tackling SDG challenges [5].

In modeling SDG-based educational quality, nonlinear delay differential equations (DDEs) are particularly effective because they capture the inherent complexities and time lags—such as delays in policy implementation or resource allocation—within educational systems . Unlike traditional ordinary differential equations, DDEs incorporate time delays τ, reflecting real-world phenomena like the postponed effects of reforms or the gradual adoption of sustainable curricula. Nonlinearity accounts for feedback loops and threshold behaviors, such as saturation in learning outcomes or complex interactions among variables like teacher training (T) and resource availability (R) [4, 7].

The utility of nonlinear DDEs has been demonstrated in various dynamic systems, including biological networks, where delays influence long-term stability and behavior [12]. For educational quality, these models simulate evolution over time, representing the inertia of institutional change. A critical aspect of these models is convergence of solutions, which ensures that quality indicators reach desired equilibria despite initial disturbances. Convergence analysis verifies the robustness of SDG interventions by checking if different initial conditions tend toward the same steady state [7].

To rigorously establish convergence in such systems, Lyapunov's direct method—also known as the second method—serves as a cornerstone, extending from ordinary differential equations to DDEs via adaptations like Razumikhin functions or Krasovskii functionals [7, 8]. This method involves constructing a positive definite scalar function whose time derivative along system trajectories is negative semidefinite, implying stability and potential convergence to equilibria. For DDEs, the Razumikhin approach modifies the standard Lyapunov function by imposing conditions that bound the function's values over the delay interval, facilitating estimates of convergence rates such as exponential, finite-time, or fixed-time [9]. In particular, Lyapunov functions from delay-free systems can be augmented with integral terms to form functionals for DDEs, preserving global asymptotic stability and ensuring solution convergence [10]. For delayed partial differential systems, exponential stability has been analyzed using similar Lyapunov-based techniques [11]. Optimization frameworks further leverage Lyapunov's method for DDEs, extending Malkin's approach to derive stability conditions [9, 11].

When assessing convergence via pairs of solutions, Lyapunov's direct method is particularly effective. By considering the differences between two solutions and defining a Lyapunov functional on these differences, one can demonstrate that the discrepancies diminish over time, leading to asymptotic convergence [13–16]. This pair-wise approach, often employing quadratic forms augmented with delay integrals, has been used to derive criteria for solution convergence in nonlinear systems, including those with stochastic perturbations or neutral terms [17]. In SDG educational models, this methodology can verify that diverse initial states—representing varying educational baselines—converge to targeted quality levels, enhancing the predictive reliability of delay-inclusive frameworks [5, 18].

Past results underscore the evolution of these techniques. Early works focused on qualitative analysis of nonautonomous nonlinear DDEs, establishing boundedness and convergence under specific growth conditions [7, 8, 13, 15]. Subsequent advancements incorporated forcing terms and biological applications, deriving global convergence criteria for perturbed models like Mackey–Glass or food-limited populations [20, 21]. In sustainability education, mathematical modeling has been linked to SDGs through teacher training programs, where pre-service educators develop modeling tasks addressing goals like clean water (SDG 6) or responsible consumption (SDG 12), fostering competencies via mixed-methods evaluations that reveal significant knowledge gains and attitudinal shifts [5, 22]. These studies build on UNESCO guidelines and emphasize interdisciplinary integration, though gaps remain in delay-inclusive models for educational dynamics.

Building on prior work ([13–15]) concerning the stability and convergence of Delay Differential Equations with forcing terms, this research proposes a third-order nonlinear DDE model for SDG-based educational quality introducing time-lagged arguments that reflect real-world lags in policy implementation and systemic response. The model is given by:

$$\begin{aligned} x'''(t) + \mu x(t-\tau)x''(t-\tau) + \alpha\left[1 - (x'(t-\tau))^2\right] = P(t-\tau) \\ P(t-\tau) = a_1 T(t-\tau) + a_2 R(t-\tau) + a_3 L(t-\tau), \end{aligned} \tag{1.1}$$

where $x(t)$ is educational quality, and $P(t)$ aggregates inputs from teacher effectiveness (T), resources (R), and learning outcomes (L). By applying Lyapunov's direct method to pairs of solutions, the study establishes uniform asymptotic stability and convergence, providing new criteria that accommodate bounded inputs and enhance the theoretical understanding of delay-inclusive educational dynamics.

2. System Formulation

System (1) becomes:

$$x'(t) = y(t) \tag{2.1}$$

$$y'(t) = z(t) \tag{2.2}$$

$$z'(t) = P(t-\tau) - \mu x(t-\tau)z(t-\tau) - \alpha\left(1 - y^2(t-\tau)\right) \tag{2.3}$$

This formulation captures:

(i) Systemic inertia and memory.
(ii) Momentum and self-regulating feedback loops.
(iii) Delayed responses to policy interventions and external influences.

3. Asymptotic Stability of Delayed Differential Equation Model for SDG-based Educational Quality

The affirmative finding of Asymptotic Stability suggests that, provided certain parameters (μ, α, and the delay τ) are within specific bounds, the education system possesses an inherent self-correcting capability that drives it back toward the SDG 4 quality target. This property considers the unforced system, obtained by setting the external input $P(t-\tau)=0$ in the original DDE (1) The trivial solution is $\mathbf{x}=\mathbf{0}$.

Theorem 3.1 (Asymptotic Stability of the Unforced System). *Assume the following condition holds:*

(i) *Positive Definiteness: A Lyapunov-Krasovskii functional $V(\mathbf{x})$ exists and is positive definite,*

(ii) *Negative Definiteness: its time derivative $\dot{V}$ along the trajectories of the unforced system (2), (3), (4) is negative definite in a neighborhood of the origin, satisfying:*

$$\frac{dV}{dt} \leq -\rho_2\left[x^2(t)+y^2(t)+z^2(t)\right]$$

for some constant $\rho_2>0$, $\mu>0$, $\alpha>0$, and the delay $\tau>0$ are fixed,

then the trivial solution $\mathbf{x}=\mathbf{0}$ of the unforced system is Asymptotically Stable:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. The proof relies on constructing a suitable Lyapunov functional $V(\mathbf{x})$ and demonstrating that the system's dynamics ensure $\dot{V}$ remains negative definite, which is the necessary condition for asymptotic stability for the trivial solution. Scalar function is defined by

$$2V = x^2+y^2+z^2+2xy+2xz+2yz+\int_{t-\tau}^{0}(\xi_1 x^2(s)+\xi_2 y^2(s)+\xi_3 z^2(s))ds, \tag{3.1}$$

can be rewritten as

$$2V = (x+y+z)^2+\xi_1(t)x^2(t-\tau)+\xi_2 y^2(t-\tau)+\xi_3 z^2(t-\tau). \geq 0. \tag{3.2}$$

Time derivatives give

$$\dot{V} = \frac{\partial V}{\partial t}x'(t)+\frac{\partial V}{\partial t}y'(t)+\frac{\partial V}{\partial t}z'(t) \tag{3.3}$$

$$\begin{aligned}
\frac{\partial V}{\partial t}x'(t) &= (x+y+z+\xi x^2(t-\tau))y \\
&= (xy+y^2+yz+\xi_1 yx^2(t-\tau)) = (x^2+2y^2+z^2)+a_1(x^2+y^2)(r(t))^2) \\
\frac{\partial V}{\partial t}y'(t) &= (x+y+z+\xi_2 y^2(t-\tau))z \\
&= (xz+z^2+yz+\xi_2 zy^2(t-\tau)) = (x^2+y^2+2z^2)+a_2(y^2+z^2)(r(t))^2)
\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial t}z'(t) &= (x+y+z+\xi_3 z^2(t-\tau))(-\mu x(t-\tau)z(t-\tau)-\alpha\left(1-y^2(t-\tau)\right)\\ &= (-x\mu x(t-\tau)z(t-\tau)-y\mu x(t-\tau)z(t-\tau)-z\mu x(t-\tau)z(t-\tau)\\ &+ \xi_3 z^2(t-\tau)\mu x(t-\tau)z(t-\tau))+\alpha x-\alpha y-\alpha z\\ &+ \alpha\xi_3 z^2(t-\tau)+y^2(t-\tau))x\mu x(t-\tau)z(t-\tau)\\ &- yy^2(t-\tau))\mu x(t-\tau)z(t-\tau)\\ &- zy^2(t-\tau))\mu x(t-\tau)z(t-\tau)+\xi_3 z^2(t-\tau)\alpha y^2(t-\tau))\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial t}z'(t) &\leq -c_1x^2z(r(t))^2-c_2\mu xyz(r(t))^2-c_3\mu xz^2(r(t))^2+c_4\xi_3\mu z^3r(t))^3\\ &+ \alpha\mid x\mid-\alpha\mid y\mid-\alpha\mid z\mid+\alpha c_5\xi_3 z^2(r(t))^2+c_6y^2x^2z(r(t))^4\\ &- c_7\mu xy^3z(r(t))^3+c_8xy^2z^2(r(t))^4+c_9\xi_3z^2\alpha y^2(r(t))^4\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial t}z'(t) &\leq -c_1x^2z(r(t))^2-c_2\mu xyz(r(t))^2-c_3\mu xz^2(r(t))^2+c_4\xi_3\mu z^3r(t))^3+\alpha\mid x\mid\\ &- \alpha\mid y\mid-\alpha\mid z\mid+\alpha c_5\xi_3 z^2(r(t))^2+c_6y^2x^2z(r(t))^4-c_7\mu xy^3z(r(t))^3\\ &- c_8xy^2z^2(r(t))^4+c_9\xi_3z^2\alpha y^2(r(t))^4\end{aligned}$$

$$\begin{aligned}\dot{V} &= (x^2+2y^2+z^2)+a_1(x^2+y^2)(r(t))^2)+(x^2+y^2+2z^2)+a_2(y^2+z^2)(r(t))^2)-\\ &- c_1x^2z(r(t))^2-c_2\mu xyz(r(t))^2-c_3\mu xz^2(r(t))^2+c_4\xi_3\mu z^3r(t))^3+\alpha\mid x\mid\\ &- \alpha\mid y\mid-\alpha\mid z\mid+\alpha c_5\xi_3z^2(r(t))^2+c_6y^2x^2z(r(t))^4-c_7\mu xy^3z(r(t))^3\\ &+ c_8xy^2z^2(r(t))^4+c_9\xi_3z^2\alpha y^2(r(t))^4\end{aligned}$$

(3.4) $$\dot{V}\leq-\rho_1(x^2+y^2+z^2)+d(x^2+y^2+z^2)(r(t))^2)$$

where

$$\begin{aligned}\rho_1 &= min\{c_1,c_2,c_3,...c_9\},c_1,c_2,c_3,...c_9,a_1,a_2>0,\ \ c_1\mu x\leq d_1>0,\\ c_2\mu y &\leq d_2>0,\ \ c_3\xi_i\alpha z\leq d_3>0,c_4\xi_3\mu z\leq d_4>0,\end{aligned}$$

(3.5) $$\dot{V}\leq-\rho_2(x^2+y^2+z^2)<0\ \ as\ \ t\longrightarrow\infty$$

□

4. CONVERGENCE OF SOLUTIONS FOR THE DELAY DIFFERENTIAL MODEL OF EDUCATIONAL QUALITY

The Lyapunov-based convergence theorem establishes uniform asymptotic stability for pairs of solutions under bounded and continuous assumptions, demonstrating that differences $(x_1(t)-x_2(t)), y_1(t)-y_2(t)), (z_1(t))-z_2(t))$ approach zero as $t \to \infty$. This theorem, augmented with integral terms in the Lyapunov functional to handle delays, extends classical results for nonlinear DDEs. For the educational sector, this implies that diverse initial conditions—reflecting varying institutional baselines or regional disparities—converge to a common high-quality equilibrium, provided delays are managed effectively. Such predictability aids policymakers in forecasting long-term impacts of SDG 4 interventions, reducing uncertainty in resource planning and enhancing accountability

Definition 4.1. *Any pair of solutions (1), (2), (3), (4) converges if* $(x_1 - x_2) \longrightarrow 0, (y_1 - y_2) \longrightarrow 0, (z_1 - z_2) \longrightarrow 0$ *as* $t \longrightarrow \infty$.

Theorem 4.1. *[Uniform Asymptotic Convergence of Solution Pairs] Let* $(x_1(t), y_1(t), z_1(t))$ *and* $(x_2(t), y_2(t), z_2(t))$ *be two solutions of the delayed system (1), assume:*

(i) *All solutions and external inputs (i.e.,* $T(t), R(t), L(t)$*) are bounded and continuous for all* $t \geq 0$, $\mu > 0, \alpha > 0$, *and* $\tau > 0$ *are fixed parameters,*
(ii) $(\mu n_1 - \alpha n_2) > 0$,
(iii) $(\xi_2 c_4 + \xi_1 c_2) > 0$, $n_1, n_2, c_1, n_2 > 0$,
(iv) *The initial functions* $x(t), y(t), z(t)$ *are continuous and bounded on* $[-\tau, 0]$.

Here $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$, $\Delta z = z_1 - z_2$, *and* $\xi_1, \xi_2, \xi_3 > 0$ *are positive constants. Then, there exists a choice of constants* ξ_1, ξ_2, ξ_3 *such that the derivative of* $V(t)$ *along the solutions satisfies:*

$$\frac{dV}{dt} \leq -\delta \left[(\Delta x(t))^2 + (\Delta y(t))^2 + (\Delta z(t))^2\right] \quad \text{for some } \delta > 0.$$

Therefore,

$$\lim_{t\to\infty} \Delta x(t) = \lim_{t\to\infty} \Delta y(t) = \lim_{t\to\infty} \Delta z(t) = 0.$$

That is, the pair of solutions converge asymptotically, and the system exhibits uniform convergence in the sense of Lyapunov.

Proof. Replacing respectively x, y, z in (1), (2), (3), (4) by the pair of solutions $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$, we introduce the Lyapunov function

$$\begin{aligned}
2V &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \\
&+ 2(x_1 - x_2)(z_1 - z_2) + 2(y_1 - y_2)(z_1 - z_2) \\
&+ \int_{t-\tau}^{0} \xi(x_1 - x_2)^2(s) + \xi_2(y_1 - y_2)^2(s) + \xi_3(z_1 - z_2)^2(s)ds \\
2V &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
&+ (x_1 - x_2)^2 + (z_1 - z_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \xi(t)(x_1 - x_2)^2(t - \tau) \\
&+ \xi_2(y_1 - y_2)^2(t - \tau) + \xi_3(z_1 - z_2)^2(t - \tau)
\end{aligned}$$

$$\begin{aligned}
V &\geq (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
&+ (x_1 - x_2)^2 + (z_1 - z_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \xi_1\tau^2(t)(x_1 - x_2)^2 \\
&+ \xi_2\tau^2(t)(y_1 - y_2)^2 + \xi_3\tau^2(t)(z_1 - z_2)^2 \\
&\geq \xi_4\Bigg((x_1 - x_2) + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\
V &\geq 0
\end{aligned}$$

Using (5), time derivatives will yield,

$$\begin{aligned}
\frac{\partial V}{\partial t}x'(t) &= (x_1 - x_2 + (y_1 - y_2) + (z_1 - z_2) + \xi_1(x_1 - x_2)^2(t - \tau))(y_1 - y_2) \\
&= (x_1 - x_2)(y_1 - y_2) + (y_1 - y_2)^2 + (z_1 - z_2)(y_1 - y_2) \\
&+ \xi_1(x_1 - x_2)^2(t - \tau))(y_1 - y_2) \\
&\leq c_1\big((x_1 - x_2)^2 + (y_1 - y_2)^2\big) + (z_1 - z_2)^2 + \xi_1 c_2\tau^2(t)(x_1 - x_2)^2(y_1 - y_2)^{1/2}
\end{aligned}$$

$$\frac{\partial V}{\partial t}y'(t) = (x_1 - x_2) + (y_1 - y_2) + (z_1 - z_2) + \xi_2(y_1 - y_2)^2(t - \tau))(z_1 - z_2)$$

$$\begin{aligned}
\frac{\partial V}{\partial t}y'(t) &= 1/2\Bigg((x_1 - x_2)^2 + (z_1 - z_2)^2) + 1/2\Bigg((y_1 - y_2)^2 + (z_1 - z_2)^2) \\
&+ (z_1 - z_2)^2 + \xi_2\tau^2(y_1 - y_2)^2(t - \tau)) \mid (z_1 - z_2) \mid \\
&\leq 1/2((x_1 - x_2)^2 + (z_1 - z_2)^2) + 1/2((y_1 - y_2)^2 + (z_1 - z_2)^2) + (z_1 - z_2)^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \xi_2 c_4 \tau^2(t)(y_1-y_2)^2(z_1-z_2)^{1/2} \\
&\leq \quad c_5(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+\xi_2 c_4 \tau^2(t)(y_1-y_2)^2(z_1-z_2)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V}{\partial z} z'(t) \quad &= \quad \Big((x_1-x_2)+(y_1-y_2)+(z_1-z_2)+\xi_3(x_1-x_2)^2(t-\tau)\Big)\Big(P(t) \\
&- \quad \mu(x_1-x_2)(t-\tau)(z_1-z_2)(t-\tau)-\alpha(1-(y_1-y_2)^2(t-\tau)\Big) \\
&= \quad (x_1-x_2)+(y_1-y_2)+(z_1-z_2)+\xi_3(z_1-z_2)^2(t-\tau))a_1T(t-\tau) \\
&+ \quad a_2R(t-\tau)+a_3L(t-\tau)+\mu(x_1-x_2)(t-\tau)(z_1-z_2)(t-\tau) \\
&- \quad \alpha\left(1-(y_1-y_2)^2(t-\tau)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V}{\partial z} z'(t) \quad &= \quad \Big((x_1-x_2)+(y_1-y_2)+(z_1-z_2)\Big)\Big(P(t)+\Big(\xi_3(x_1-x_2)^2(t-\tau)\Big)P(t) \\
&- \quad \Big(\mu(x_1-x_2)(t-\tau)(z_1-z_2)(t-\tau)\Big)((x_1-x_2)+(y_1-y_2)+(z_1-z_2)) \\
&- \quad \Big(\alpha(1-(y_1-y_2)^2(t-\tau)\Big)((x_1-x_2)+(y_1-y_2)+(z_1-z_2)) \\
&= \quad \Big((x_1-x_2)+(y_1-y_2)+(z_1-z_2)\Big) \\
&\cdot \quad \Big(a_1T(t-\tau)+a_2R(t-\tau)+a_3L(t-\tau)\Big) \\
&- \quad \mu n_1\Big((r(t))^2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\Big)-\alpha\Big((x_1-x_2)^2 \\
&+ \quad (y_1-y_2)^2+(z_1-z_2)^2)\Big)^{1/2}+\alpha n_2(r(t))^2 \\
&\cdot \quad \Big((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)\Big) \\
&= \quad a_4(r(t))^3\Big((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)\Big)^{1/2} \\
&- \quad \mu n_1\Big((r(t))^2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\Big)-\alpha\Big((x_1-x_2)^2
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad (y_1 - y_2)^2 + (z_1 - z_2)^2)\Big)^{1/2} + \alpha n_2 (r(t))^2 \\
\cdot &\quad \Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V}{\partial z} z'(t) &\leq -(\mu n_1 - \alpha n_2)(r(t))^2 \Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\Big) \\
&+ a_4((r(t))^3 - \alpha)\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{V} &= c_1 \Big((x_1 - x_2)^2 + (y_1 - y_2)^2) + (z_1 - z_2)^2 + \xi_1 c_2 \tau^2(t)(x_1 - x_2)^2 (y_1 - y_2)^{1/2} \\
&+ c_5 (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \xi_2 c_4 \tau^2(t)(y_1 - y_2)^2 (z_1 - z_2)^{1/2} \\
&- (\mu n_1 - \alpha n_2)(r(t))^2 \Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\Big) \\
&+ a_4((r(t))^3 - \alpha)\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{V} &\leq -((\mu n_1 - \alpha n_2)(r(t))^2 - c_1 - c_5)\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\Big) \\
&+ ((\xi_2 c_4 + \xi_1 c_2)\tau^2(t) + a_4((r(t))^3 - \alpha)\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)\Big)^{1/2}
\end{aligned}
$$

$$
\dot{V} \leq -n\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\Big), \quad n > 0
$$

The above inequality implies that $V(t)$ decreases to zero, indicating the convergence of the pair of solutions, i.e $(x_1 - x_2) \to 0, (y_1 - y_2) \to 0, (z_1 - z_2) \to 0$ as $t \to \infty$, validating the theorem (2). □

5. Simulation and Numerical Verification

Simulations via Mathematica code confirm that for small values of delay terms τ, solutions converge (stable) whereas for larger values τ, instability and divergence occur.

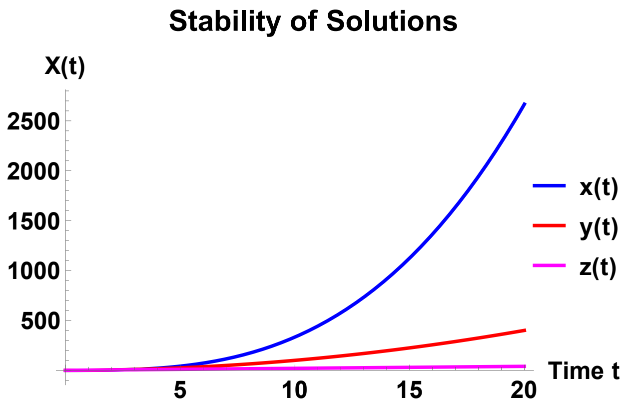

Fig. 1. The plot of the stability in SDG Model at $P(t - \tau) = 0$.

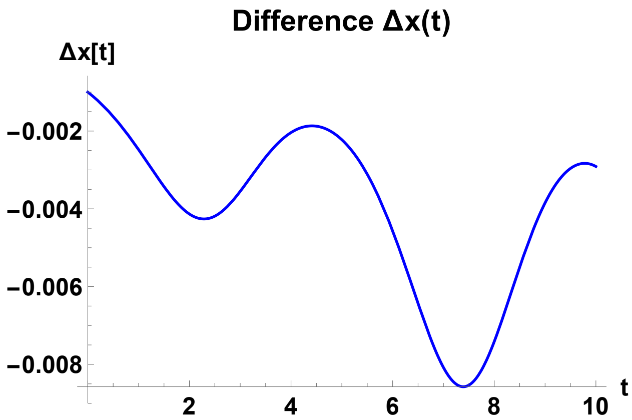

Fig. 2. Convergence of $(x_1 - x_2)$ at $\mu = 1, \alpha = 1, \tau = 0.1, P = 3$.

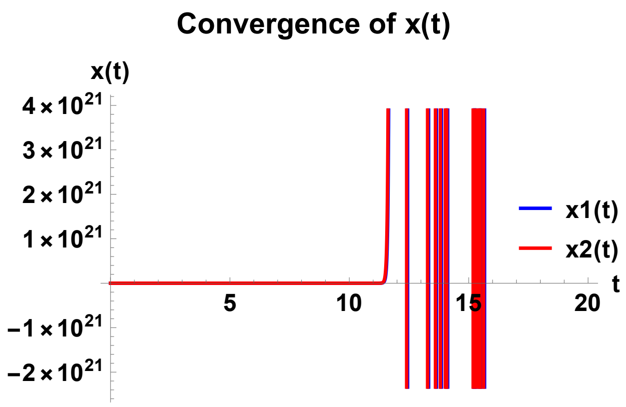

Fig. 3. Convergence of x_1, x_2 at $\mu = 1, \alpha = 0.1, \tau = 1, P = 3$.

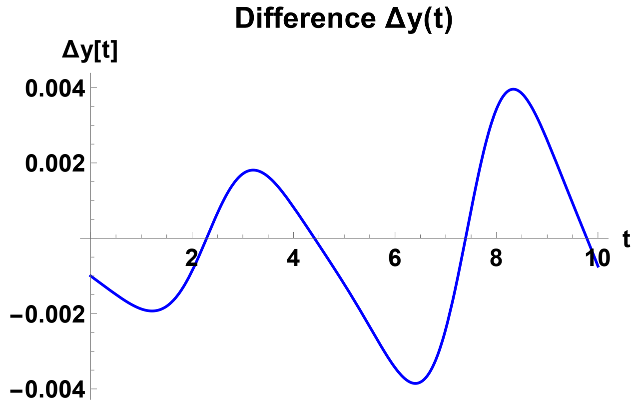

Fig. 4. Convergence of $(y_1 - y_2)$ at $\mu = 1, alpha = 1, \tau = 0.1, P = 3$.

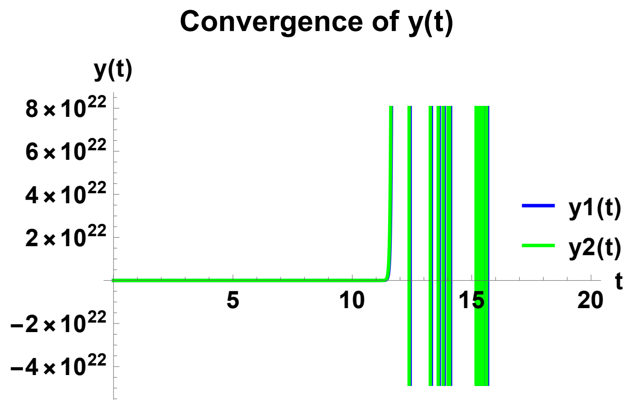

Fig. 5. Convergence of y_1, y_2 at $\mu = 1, \alpha = 1, \tau = 0.1, P = 3$.

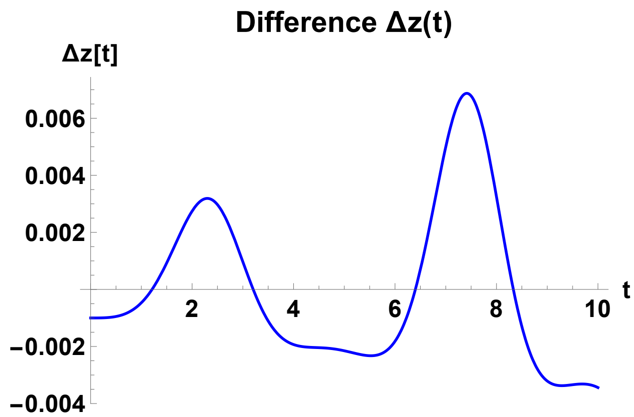

Fig. 6. Convergence of (z_1, z_2) at $\mu = 1, \alpha = 1, \tau = 0.1, P = 3$.

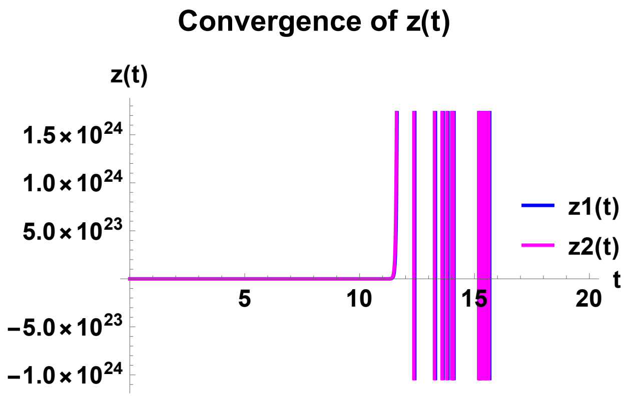

Fig. 7. Convergence of z_1, z_2 at $\mu = 1, \alpha = 1, \tau = 0.1, P = 3$.

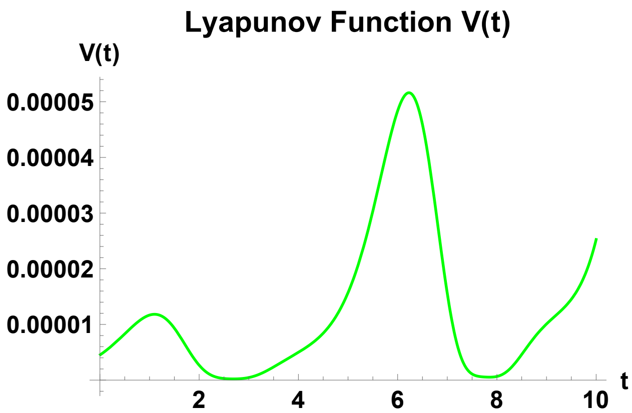

Fig. 8. The Lyapunov function experiences energy decay.

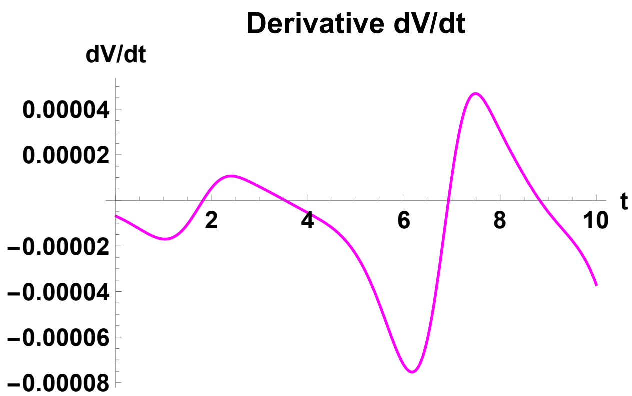

Fig. 9. Lyapunov derivative satisfies Convergence at $\tau = 0.1, \mu = 0.5, P = 1$.

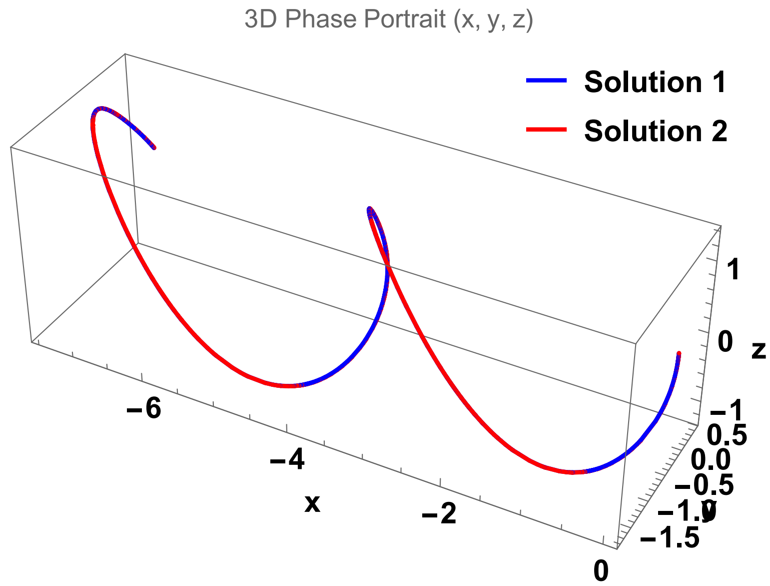

Fig. 10. Phase Portrait of x,y,z dimension at $\tau = 0.1, \mu = 0.5, P = 1$.

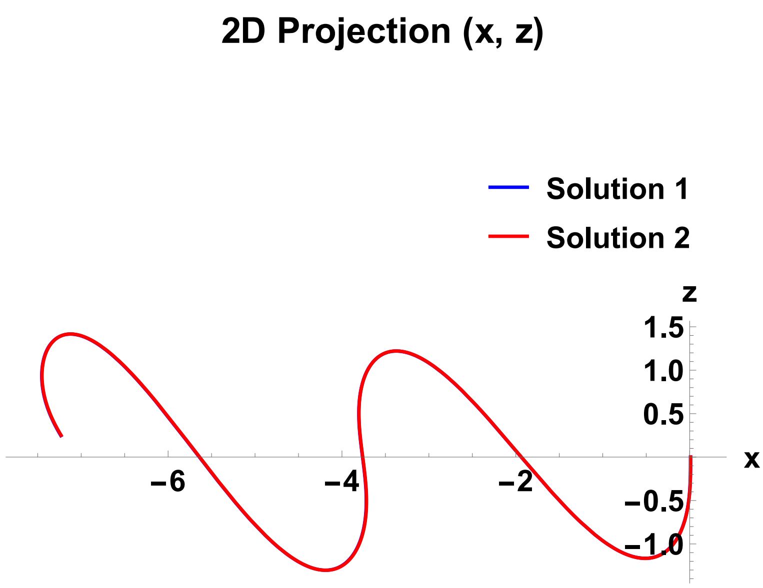

Fig. 11. Phase Portrait of x,z trajectories at $\tau = 0.1, \mu = 0.5, P = 1$.

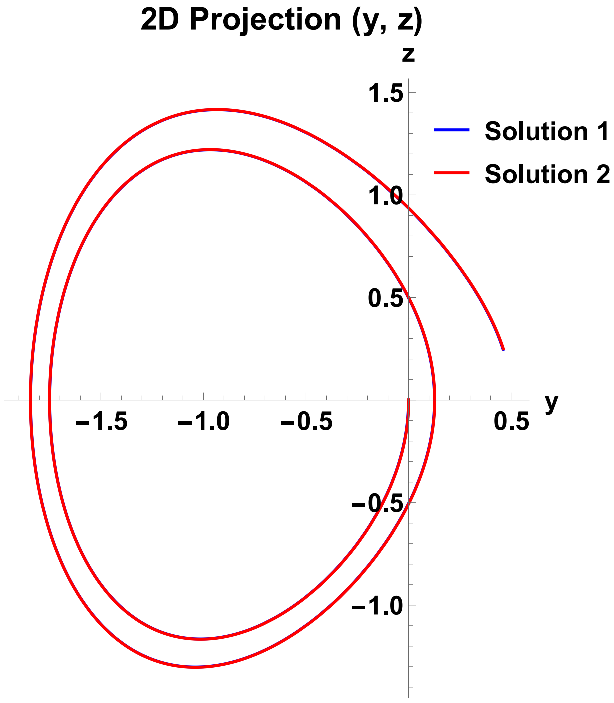

Fig. 12. Phase Portrait of y,z trajectories center with curves at $\tau = 0.1, \mu = 0.5, P = 1.$.

Convergence of Pair of Solutions

```
parameters = {μ = 1, α = 1, τ = 1, P = 3};
sol1 = NDSolve[{x1'[t] == y1[t], y1'[t] == z1[t], z1'[t] == P - μ x1[t - τ] z1[t - τ] - α (1 - y1[t - τ]^2),
    x1[t /; t ≤ 0] == 1, y1[t /; t ≤ 0] == 0, z1[t /; t ≤ 0] == 0}, {x1, y1, z1}, {t, 0, 20},
   MaxSteps → ∞, Method → {"StiffnessSwitching"}];
sol2 = NDSolve[{x2'[t] == y2[t], y2'[t] == z2[t], z2'[t] == P - μ x2[t - τ] z2[t - τ] - α (1 - y2[t - τ]^2),
    x2[t /; t ≤ 0] == 1.2, y2[t /; t ≤ 0] == 0.2, z2[t /; t ≤ 0] == 0.2}, {x2, y2, z2}, {t, 0, 20},
   MaxSteps → ∞, Method → {"StiffnessSwitching"}];
Plot[{x1[t] /. sol1, x2[t] /. sol2}, {t, 0, 20}, PlotLegends → Placed[{"x1(t)", "x2(t)"}, 2],
 PlotLabel -> "Convergence of x(t)", PlotStyle -> {{Thick, Blue}, {Thick, Red}}, AxesLabel → {"t", "x(t)"},
 LabelStyle -> {FontFamily -> "Helvetica", 14, GrayLevel[0], Bold}]
Plot[{y1[t] /. sol1, y2[t] /. sol2}, {t, 0, 20}, PlotLegends → Placed[{"y1(t)", "y2(t)"}, 2],
 PlotLabel -> "Convergence of y(t)", PlotStyle -> {{Thick, Blue}, {Thick, Green}}, AxesLabel → {"t", "y(t)"},
 LabelStyle -> {FontFamily -> "Helvetica", 14, GrayLevel[0], Bold}]
Plot[{z1[t] /. sol1, z2[t] /. sol2}, {t, 0, 20}, PlotLegends → Placed[{"z1(t)", "z2(t)"}, 2],
 PlotLabel -> "Convergence of z(t)", PlotStyle -> {{Thick, Blue}, {Thick, Magenta}}, AxesLabel → {"t", "z(t)"},
 LabelStyle -> {FontFamily -> "Helvetica", 14, GrayLevel[0], Bold}]
diffx = Plot[(x1[t] - x2[t]) /. {sol1, sol2}, {t, 0, 20}, PlotLabel → "Difference Δx(t)"]
diffy = Plot[(y1[t] - y2[t]) /. {sol1, sol2}, {t, 0, 20}, PlotLabel → "Difference Δy(t)"]
diffz = Plot[(z1[t] - z2[t]) /. {sol1, sol2}, {t, 0, 20}, PlotLabel → "Difference Δz(t)"]
GraphicsGrid[{{diffx, diffy, diffz}}]
```

Fig. 13. Wolfram code shows convergence of pair of solutions.

```
Convergence, Lyapunov Function, Phase Portraits
parameters = {μ = 0.01, α = 1, τ = 0.01, P = 0, ξ1 = 1, ξ2 = 1, ξ3 = 1};
sol1 = NDSolve[{x1'[t] == y1[t], y1'[t] == z1[t], z1'[t] == P - μ x1[t - τ] z1[t - τ] - α (1 - y1[t - τ]^2),
    x1[t /; t ≤ 0] == 0.01, y1[t /; t ≤ 0] == 0, z1[t /; t ≤ 0] == 0}, {x1, y1, z1}, {t, 0, 10}, MaxSteps → ∞,
   Method → {"StiffnessSwitching"}, PrecisionGoal → 8, AccuracyGoal → 8];
sol2 = NDSolve[{x2'[t] == y2[t], y2'[t] == z2[t], z2'[t] == P - μ x2[t - τ] z2[t - τ] - α (1 - y2[t - τ]^2),
    x2[t /; t ≤ 0] == 0.011, y2[t /; t ≤ 0] == 0.001, z2[t /; t ≤ 0] == 0.001}, {x2, y2, z2}, {t, 0, 10}, MaxSteps → ∞,
   Method → {"StiffnessSwitching"}, PrecisionGoal → 8, AccuracyGoal → 8];
Δx[t_] := (x1[t] /. First[sol1]) - (x2[t] /. First[sol2]);
Δy[t_] := (y1[t] /. First[sol1]) - (y2[t] /. First[sol2]);
Δz[t_] := (z1[t] /. First[sol1]) - (z2[t] /. First[sol2]);
V[t_ /; t ≥ τ] := (1 / 2) * (Δx[t]^2 + Δy[t]^2 + Δz[t]^2 + 2 Δx[t] Δy[t] + 2 Δx[t] Δz[t] + 2 Δy[t] Δz[t]
     + NIntegrate[ξ1 Δx[s]^2 + ξ2 Δy[s]^2 + ξ3 Δz[s]^2, {s, t - τ, t}, AccuracyGoal → 8]);
phase3D = ParametricPlot3D[{{x1[t], y1[t], z1[t]} /. First[sol1], {x2[t], y2[t], z2[t]} /. First[sol2]},
   {t, 0, 10}, PlotLegends → Placed[{"Solution 1", "Solution 2"}, 2], PlotLabel → "3D Phase Portrait (x, y, z)",
   AxesLabel → {"x", "y", "z"}, PlotStyle → {Blue, Red}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
projXY = ParametricPlot[{{x1[t], y1[t]} /. First[sol1], {x2[t], y2[t]} /. First[sol2]}, {t, 0, 10},
   PlotLegends → Placed[{"Solution 1", "Solution 2"}, 2], PlotLabel → "2D Projection (x, y)",
   AxesLabel → {"x", "y"}, PlotStyle → {Blue, Red}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
projXZ = ParametricPlot[{{x1[t], z1[t]} /. First[sol1], {x2[t], z2[t]} /. First[sol2]}, {t, 0, 10},
   PlotLegends → Placed[{"Solution 1", "Solution 2"}, 2], PlotLabel → "2D Projection (x, z)",
   AxesLabel → {"x", "z"}, PlotStyle → {Blue, Red}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
projYZ = ParametricPlot[{{y1[t], z1[t]} /. First[sol1], {y2[t], z2[t]} /. First[sol2]}, {t, 0, 10},
   PlotLegends → Placed[{"Solution 1", "Solution 2"}, 2], PlotLabel → "2D Projection (y, z)",
   AxesLabel → {"y", "z"}, PlotStyle → {Blue, Red}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
diffx = Plot[Δx[t], {t, 0, 10}, PlotLabel → "Difference Δx(t)", PlotStyle → Blue,
   AxesLabel → {"Δx[t]", "t"}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
diffy = Plot[Δy[t], {t, 0, 10}, PlotLabel → "Difference Δy(t)", PlotStyle → Blue, AxesLabel → {"Δy[t]", "t"},
     LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}] diffz = Plot[Δz[t], {t, 0, 10},
    PlotLabel → "Difference Δz(t)", PlotStyle → Blue, AxesLabel → {"Δz[t]", "t"},
    LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
vPlot = Plot[V[t], {t, τ, 10}, PlotLabel → "Lyapunov Function V(t)", PlotStyle → Green, PlotRange → All,
   AxesLabel → {"V(t)", "t"}, LabelStyle → {FontFamily → "Helvetica", 14, GrayLevel[0], Bold}];
dVdt = Plot[D[V[t], t], {t, τ, 10}, PlotLabel → "Derivative dV/dt", PlotStyle → Purple, PlotRange → All]
```

Fig. 14. Wolfram code shows shows asymptotic analysis of the pair of solutions.

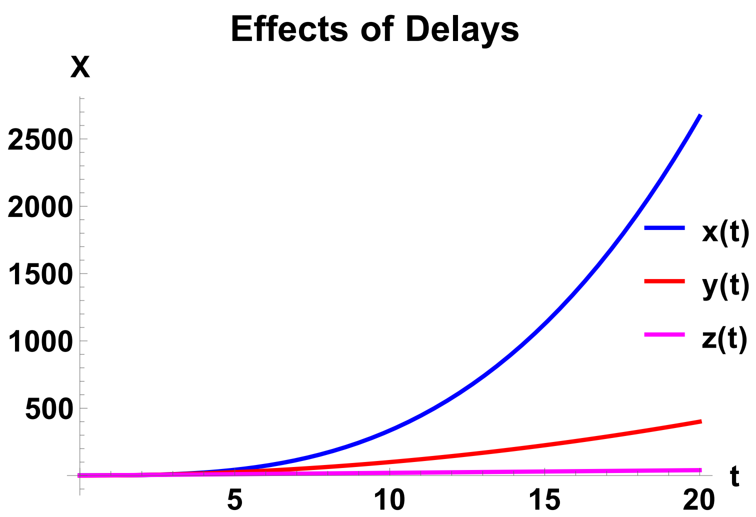

Fig. 15. Effect of delays at $\tau = 1$

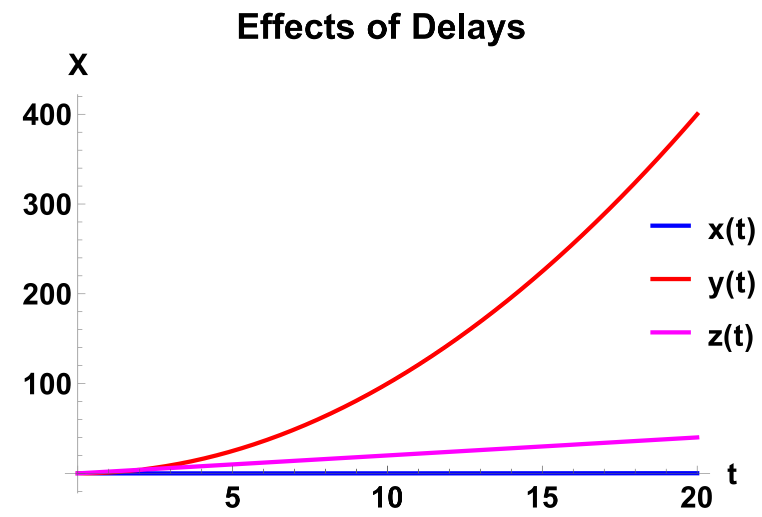

Fig. 16. Effect of delays at $\tau = 0.03$

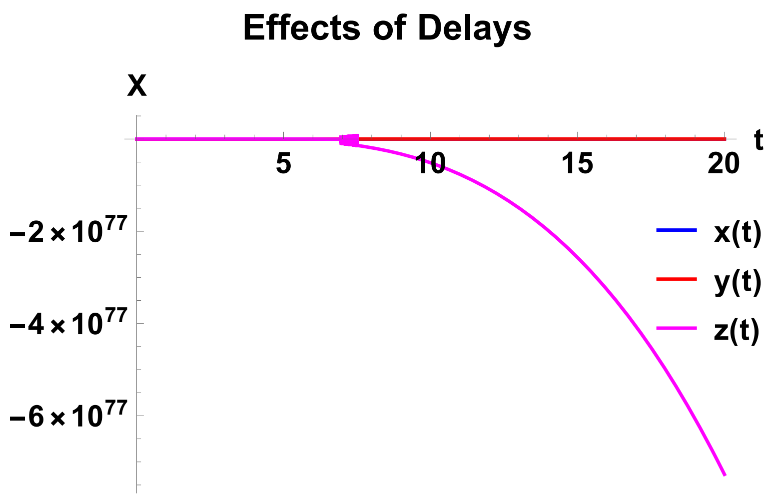

Fig. 17. Effect of delays at $\tau = 0.01$

6. Results and Discussion

The numerical analysis in Figure 1 shows that With $\frac{dV}{dt} \leq 0$, $V(t)$ positive definite where P(t) = 0, standard results for delay differential equations imply asymptotic stability: $x(t), y(t), z(t) \rightarrow 0$ as $t \rightarrow \infty$. Figures 2-6 visually confirm the convergence of the pair of solutions. The trajectories for educational quality and its derivatives $(x_1(t), x_2(t)$, etc.) remain close due to small initial differences. With specific parameter values $(\tau = 0.1, 0.03, 0.05, \mu = 0.5, P = 1)$, the solutions exhibit stable, bounded behavior. This damping effect is attributed to the negative term, $-\mu x(t-\tau)z(t-\tau)$, which helps to reduce the growth of $z(t)$ (acceleration), thereby promoting convergence.

Figure 7 and 8 represent analysis of the Lyapunov function $V(t)$ and its derivative,

Phase Portraits in Figures 9a, b, c illustrate the dynamics of the trajectories in three-dimensional (x, y, z) space. Starting from slightly different initial points, the trajectoriesconverge spirally to a similar path, confirming that the differences $\Delta x, \Delta y, \Delta z$ tend to zero. Two-dimensional projections further show that convergence manifests as overlapping curves for the (x, y), (x, z), and (y, z) pairs.

The analysis of delays (Figures 14-17) indicates that systems with small delays exhibit stable behavior where solutions remain bounded and the pair of solutions converges asymptotically $(\Delta(t) \to 0)$. Specifically, short delays ($\tau < 0.1$) enhance the robustness of quality improvement, allowing for rapid impacts from teacher training (τ_T) or resource deployment (τ_R). Conversely, medium-to-large delays ($\tau > 1$) lead to instability and divergence, highlighting the need for efficient policy implementation.

7. Summary and Conclusion

The comprehensive analysis encompassing theoretical convergence proofs via the Lyapunov's direct method, numerical verification using Mathematica, and qualitative analysis of delay magnitudes provide significant insights for the educational sector.

The Lyapunov-based convergence theorem formally establishes the uniform asymptotic stability of solution pairs under continuous and bounded assumptions, proving that the solution differences $(\Delta x, \Delta y, \Delta z)$ approach zero over time. For SDG 4, this means that varied initial educational conditions, such as **regional disparities or institutional baselines, will converge to a shared high-quality equilibrium, provided that the systemic delays are effectively managed. This predictability is valuable for policymakers in forecasting the long-term impacts of SDG 4 interventions, aiding in resource planning and enhancing accountability.

The numerical verification of the Lyapunov derivative $\frac{dV}{dt} \leq -\delta[(\Delta x(t))^2 + (\Delta y(t))^2 + (\Delta z(t))^2]$ confirms its negative definiteness within stable regimes. This approach can be utilized as a **tool for assessing policy robustness**, allowing for simulating scenarios, such as the effects of a delay in teacher training (τ_T), to inform evidence-based reforms.

Simulations confirm that small delays ($\tau < 0.1$) drive rapid convergence, while medium-to-large delays ($\tau > 1$) induce oscillations or divergence. Phase portraits clearly show stable trajectories merging for small delays, contrasting with chaotic divergence for larger ones, underscoring the destabilizing influence of prolonged lags. These findings highlight the necessity of minimizing implementation delays to meet SDG 4 targets, as short lags (e.g., ≈ 0.03) ensure prompt responses to external inputs $P(t)$, thereby optimizing investments in teachers (T), resources (R), and learning outcomes (L).

At the learner level, the predicted stable convergence supports the development of resilient educational pathways, ensuring that initial disadvantages (e.g., in access or prior knowledge) are reduced over time. Decreased delays in resource allocation and learning feedback lead to smoother quality momentum ($y(t)$) and acceleration ($z(t)$), fostering consistent progress and lowering dropout risks. Ultimately, this translates to more equitable access to quality education, consistent with SDG 4's emphasis on inclusivity.

In conclusion, this model advances both the theoretical and applied dimensions of educational modeling by offering a delay-aware framework for policy design that improves sector efficiency and learner success.

CONFLICT OF INTEREST

There is no Conflict of Interest

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