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ON TWO NEW THEOREMS ON CONVEX INTEGRAL INEQUALITIES

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ABSTRACT. This paper presents two new integral inequalities involving a convex function and an auxiliary function satisfying mild analytical conditions. Illustrative examples, including cases on the unit interval, are provided to demonstrate the applicability of the obtained results.

1. INTRODUCTION

Convex analysis is fundamental to various branches of mathematics. Convex functions, in particular, form the theoretical basis for many key results, such as major inequalities, duality principles and stability theorems. A precise definition is stated below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ be such that $a < b$. A function $f : [a, b] \to \mathbb{R}$ is called convex (on $[a, b]$) if, for any pair of points $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality is satisfied:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

One of the most notable results in this area is the Hermite-Hadamard integral inequality. A precise formulation is given below. Let $a, b \in \mathbb{R}$ with $a < b$, and

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$f : [a, b] \to \mathbb{R}$ be a convex function. Then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This classical result therefore establishes a fundamental relationship between the value of a convex function at the midpoint of an interval and its mean integral, as well as its values at the endpoints. It forms the basis for numerous generalizations and applications in analysis. Further information on convex functions and related inequalities can be found in [1–14].

In this paper, we contribute to the convex analysis by establishing two new integral inequalities involving a convex function. A distinctive aspect of these results lies in their dependence on an auxiliary function φ, which is assumed to satisfy mild conditions. In particular, the first inequality takes the form

$$\int_a^b f(x)dx \leq \int_a^b w(x)f(x)dx,$$

where $f : [a, b] \to \mathbb{R}$ denotes the convex function of interest, and $w : [a, b] \to [0, +\infty)$ denotes a weight function determined by φ. The second inequality is more sophisticated, involving three integrals. Several illustrative examples are provided to demonstrate the applicability of the obtained results, including specific cases on the unit interval $[0, 1]$. The proofs primarily rely on integral operations, such as changing the variables and integrating by parts, together with the convexity inequality expressed in Equation (1.1).

The remainder of this paper is organized as follows: In Section 2, we present the main theoretical results and corresponding proofs. Finally, Section 3 offers concluding remarks and potential directions for future work.

2. Results

2.1. First theorem.

The theorem below presents our first convex integral inequality, followed by its proof and some examples. It is mainly based on an appropriate change of variables and the inequality in Equation (1.1).

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$ and $0 \in [a, b]$, $f : [a, b] \to \mathbb{R}$ be a convex function with $f(0) = 0$, and $\varphi : [a, b] \to [0, +\infty)$ be a differentiable increasing function such that $\varphi(a) = a$, $\varphi(b) = b$, and, for any $x \in [a, b]$, $\varphi(x) \leq x$. Then the* *following inequality is satisfied:*

$$\int_a^b f(x)dx \leq \int_a^b \frac{\varphi'(x)\varphi(x)}{x} f(x)dx,$$

provided that the integrals involved converge.

Proof. Making the change of variables $y = \varphi(x)$ with $\varphi^{-1}(a) = a$ and $\varphi^{-1}(b) = b$, and applying the convex inequality to f in Equation (1.1) with $\lambda = \varphi(x)/x \in [0,1]$ and $f(0) = 0$, we get

$$\begin{aligned}
\int_a^b f(y)dy &= \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \varphi'(x) f(\varphi(x))dx \\
&= \int_a^b \varphi'(x) f(\varphi(x))dx \\
&= \int_a^b \varphi'(x) f\left(x\frac{\varphi(x)}{x} + 0 \times \left(1 - \frac{\varphi(x)}{x}\right)\right) dx \\
&\leq \int_a^b \varphi'(x) \left(\frac{\varphi(x)}{x} f(x) + \left(1 - \frac{\varphi(x)}{x}\right) f(0)\right) dx \\
&= \int_a^b \frac{\varphi'(x)\varphi(x)}{x} f(x)dx. \qquad (2.1)
\end{aligned}$$

Therefore, making a minor change in notation of the first integral, we obtain

$$\int_a^b f(x)dx \leq \int_a^b \frac{\varphi'(x)\varphi(x)}{x} f(x)dx.$$

This completes the proof. □

Theorem 2.1 is interesting because we do not necessarily have, for any $x \in [a,b]$,

$$\frac{\varphi'(x)\varphi(x)}{x} \geq 1.$$

The obtained inequality is thus not trivial.

Some examples of Theorem 2.1 are given below.

- Setting $a, b \in \mathbb{R}$, and

$$\varphi(x) = a + (b-a)\left(\frac{x-a}{b-a}\right)^p$$

with $p \geq 1$, the assumptions required on φ in Theorem 2.1 are satisfied. For a convex function $f : [a,b] \to \mathbb{R}$ with $f(0) = 0$, Theorem 2.1 gives

$$\int_a^b f(x)dx \leq p \int_a^b \frac{1}{x} \left(a + (b-a) \left(\frac{x-a}{b-a} \right)^p \right) \left(\frac{x-a}{b-a} \right)^{p-1} f(x)dx.$$

In particular, setting $a = 0$ and $b = 1$, we get

$$\int_0^1 f(x)dx \leq p \int_0^1 x^{2(p-1)} f(x)dx.$$

To be more specific, taking $f(x) = e^x - 1$ and $p = 2$, we check that

$$\int_0^1 (e^x - 1)dx \approx 0.7182 \leq 0.7699 \approx 2 \int_0^1 x^2 (e^x - 1)dx.$$

- Setting $a, b \in \mathbb{R}$, and

$$\varphi(x) = x - \lambda(x-a)(b-x)$$

with $\lambda \in [0, 1/(b-a))$, the assumptions required on φ in Theorem 2.1 are satisfied. For a convex function $f : [a,b] \to \mathbb{R}$ with $f(0) = 0$, Theorem 2.1 gives

$$\int_a^b f(x)dx \leq \int_a^b \frac{1}{x} (x - \lambda(x-a)(b-x)) (1 - \lambda(a+b-2x)) f(x)dx.$$

In particular, setting $a = 0$ and $b = 1$, we get

$$\int_0^1 f(x)dx \leq \int_0^1 (1 - \lambda(1-x)) (1 - \lambda(1-2x)) f(x)dx.$$

To be more specific, taking $f(x) = e^x - 1$ and $\lambda = 1/2$, we check that

$$\int_0^1 (e^x - 1)dx \approx 0.7182 \leq 0.7470$$
$$\approx \int_0^1 \left(1 - \frac{1}{2}(1-x) \right) \left(1 - \frac{1}{2}(1-2x) \right) (e^x - 1)dx.$$

These two numerical examples demonstrate how precise the result is.

To the best of the knowledge of the author, these are new convex integral inequalities in the literature.

Other examples of functions φ depending on exponential, logarithmic or trigonometric functions can be considered.

The integral inequality described below is a consequence of Theorem 2.1 combined with the Hölder integral inequality. For any $p > 1$ and q satisfying $1/p+1/q = 1$, the following inequality is satisfied:

$$\int_a^b f(x)dx \leq \min\left\{\left(\int_a^b \left(\frac{\varphi'(x)\varphi(x)}{x}\right)^q dx\right)^{1/q}, 1\right\}\left(\int_a^b f^p(x)dx\right)^{1/p}.$$

Depending on the definitions of f and φ, the Hölder integral inequality can thus be improved; it is the case if

$$\int_a^b \left(\frac{\varphi'(x)\varphi(x)}{x}\right)^q dx \leq 1.$$

Further inequalities of a similar nature can be derived by considering alternative weight function constructions based on their interaction.

To conclude this section, note that if f is concave rather than convex, the inequality in Theorem 2.1 is reversed.

2.2. **Second theorem.** The theorem below presents our second convex integral inequality, followed by its proof and some examples. It is mainly based on Theorem 2.1 and a suitable integration by parts.

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $b > a$ and $0 \in [a, b]$, $f : [a, b] \to \mathbb{R}$ be a convex function with $f(0) = 0$, and $\varphi : [a, b] \to \mathbb{R}$ be a differentiable increasing function such that $\varphi(a) = a$, $\varphi(b) = b$, and, for any $x \in [a, b]$, $\varphi(x) \leq x$. Then the following inequality is satisfied:*

$$\int_a^b f(x)dx + \frac{1}{2}\int_a^b \varphi^2(x)\frac{f'(x)}{x}dx \leq \frac{1}{2}(bf(b) - af(a)) + \frac{1}{2}\int_a^b \varphi^2(x)\frac{f(x)}{x^2}dx,$$

provided that the integrals involved converge.

Proof. It follows from Theorem 2.1 that

$$\int_a^b f(x)dx \leq \int_a^b \frac{\varphi'(x)\varphi(x)}{x} f(x)dx. \tag{2.2}$$

The rest of the proof involves developing the integral on the right-hand side. An appropriate integration by parts using $[\varphi^2(x)]' = 2\varphi(x)\varphi'(x)$, $\varphi(a) = a$ and $\varphi(b) = b$

gives

$$\int_a^b \frac{\varphi'(x)\varphi(x)}{x} f(x)dx = \left[\frac{1}{2}\varphi^2(x)\frac{1}{x}f(x)\right]_a^b - \frac{1}{2}\int_a^b \varphi^2(x)\frac{xf'(x)-f(x)}{x^2}dx$$

$$= \frac{1}{2}\left(\varphi^2(b)\frac{1}{b}f(b) - \varphi^2(a)\frac{1}{a}f(a)\right) - \frac{1}{2}\int_a^b \varphi^2(x)\frac{f'(x)}{x}dx \tag{2.3}$$

$$+ \frac{1}{2}\int_a^b \varphi^2(x)\frac{f(x)}{x^2}dx$$

$$= \frac{1}{2}(bf(b) - af(a)) - \frac{1}{2}\int_a^b \varphi^2(x)\frac{f'(x)}{x}dx + \frac{1}{2}\int_a^b \varphi^2(x)\frac{f(x)}{x^2}dx. \tag{2.4}$$

It follows from Equations (2.2) and (2.3) that

$$\int_a^b f(x)dx + \frac{1}{2}\int_a^b \varphi^2(x)\frac{f'(x)}{x}dx \leq \frac{1}{2}(bf(b) - af(a)) + \frac{1}{2}\int_a^b \varphi^2(x)\frac{f(x)}{x^2}dx.$$

This completes the proof. □

Multiplying by 2, an equivalent formulation of Theorem 2.2 is

$$2\int_a^b f(x)dx + \int_a^b \varphi^2(x)\frac{f'(x)}{x}dx \leq (bf(b) - af(a)) + \int_a^b \varphi^2(x)\frac{f(x)}{x^2}dx.$$

Upon analyzing the proof in detail, we can see that Theorems 2.1 and 2.2 are in fact equivalent. It can be noted that, unlike Theorem 2.1, Theorem 2.2 does not involve φ'. Additionally, a distinctive feature of Theorem 2.2 is its use of the derivative f', which is rather uncommon in the context of convex integral inequalities.

Some examples of Theorem 2.2 are given below.

- Setting $a, b \in \mathbb{R}$, and

$$\varphi(x) = a + (b-a)\left(\frac{x-a}{b-a}\right)^p$$

with $p \geq 1$, the assumptions required on φ in Theorem 2.2 are satisfied. For a convex function $f : [a, b] \to \mathbb{R}$ with $f(0) = 0$, Theorem 2.2 gives

$$\int_a^b f(x)dx + \frac{1}{2}\int_a^b \left(a + (b-a)\left(\frac{x-a}{b-a}\right)^p\right)^2 \frac{f'(x)}{x}dx$$
$$\leq \frac{1}{2}(bf(b) - af(a)) + \frac{1}{2}\int_a^b \left(a + (b-a)\left(\frac{x-a}{b-a}\right)^p\right)^2 \frac{f(x)}{x^2}dx.$$

In particular, setting $a = 0$ and $b = 1$, we get

$$\int_0^1 f(x)dx + \frac{1}{2}\int_0^1 x^{2p-1} f'(x)dx \leq \frac{1}{2} f(1) + \frac{1}{2}\int_0^1 x^{2(p-1)} f(x)dx.$$

To be more specific, taking $f(x) = e^x - 1$ and $p = 2$, we check that

$$\int_0^1 (e^x - 1)dx + \frac{1}{2}\int_0^1 x^3 e^x dx = 1 \leq 1.0516$$
$$\approx \frac{1}{2}(e-1) + \frac{1}{2}\int_0^1 x^2(e^x - 1)dx.$$

- Setting $a, b \in \mathbb{R}$, and

$$\varphi(x) = x - \lambda(x-a)(b-x)$$

with $\lambda \in [0, 1/(b-a))$, the assumptions required on φ in Theorem 2.2 are satisfied. For a convex function $f : [a,b] \to \mathbb{R}$ with $f(0) = 0$, Theorem 2.2 gives

$$\int_a^b f(x)dx + \frac{1}{2}\int_a^b (x - \lambda(x-a)(b-x))^2 \frac{f'(x)}{x} dx$$
$$\leq \frac{1}{2}(bf(b) - af(a)) + \frac{1}{2}\int_a^b (x - \lambda(x-a)(b-x))^2 \frac{f(x)}{x^2} dx.$$

In particular, setting $a = 0$ and $b = 1$, we get

$$\int_0^1 f(x)dx + \frac{1}{2}\int_0^1 x\left(1 - \lambda(1-x)\right)^2 f'(x)dx$$
$$\leq \frac{1}{2} f(1) + \frac{1}{2}\int_0^1 \left(1 - \lambda(1-x)\right)^2 f(x)dx.$$

To be more specific, taking $f(x) = e^x - 1$ and $\lambda = 1/2$, we check that

$$\int_0^1 (e^x - 1)dx + \frac{1}{2}\int_0^1 x\left(1 - \frac{1}{2}(1-x)\right)^2 e^x dx \approx 1.09328$$
$$\leq 1.1220 \approx \frac{1}{2}(e-1) + \frac{1}{2}\int_0^1 \left(1 - \frac{1}{2}(1-x)\right)^2 (e^x - 1)dx.$$

These two numerical examples demonstrate how precise the result is.

To the best of the knowledge of the author, these are new convex integral inequalities in the literature.

To conclude this section, note that if f is concave rather than convex, the inequality in Theorem 2.2 is reversed.

3. Conclusion

In this paper, we have established two new integral inequalities involving a convex function and an auxiliary function φ satisfying mild analytical conditions. Some examples illustrate the applicability and robustness of the proposed approach in different settings. Future research may focus on exploring analogous inequalities for classes of s-convex or logarithmically convex functions, as well as on developing multidimensional and fractional integral versions of the results.

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