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A NEW HARDY-HILBERT-TYPE INTEGRAL INEQUALITY INVOLVING AN EXPONENTIAL-POWER KERNEL FUNCTION

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ABSTRACT. This article investigates a new Hardy-Hilbert-type integral inequality involving a kernel function combining exponential decay and a singular power term. The main novelty lies in this feature, as well as in the fact that the resulting constant factor is expressed explicitly in terms of the gamma function. Furthermore, we demonstrate the versatility of the proposed inequality by deriving two additional integral inequalities. The proofs are presented in detail and can be followed step by step.

1. INTRODUCTION

Integral inequalities constitute a fundamental component of mathematical analysis. They are essential for exploring the properties of functions, obtaining estimates for the solutions of differential and integral equations, and establishing bounds across many branches of mathematics. These inequalities connect the local characteristics of functions, such as convexity, monotonicity, and differentiability, to their global integral behavior.

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Classical examples include the Cauchy-Schwarz integral inequality, which provides bounds for inner products and integrals of product functions (see, for example, [10]). The Hölder integral inequality generalizes this framework to L^p spaces and is fundamental to functional analysis (see, for example, [14]). The Jensen integral inequality links convexity with integration (see, for example, [11]), illustrating how the integral of a convex function compares with the function evaluated at the mean. The Grönwall integral inequality offers crucial estimates for the solutions of differential equations and serves as a fundamental tool in stability analysis (see, for example, [7]). Similarly, the Poincaré integral inequality relates the norm of a function to the norm of its derivatives on bounded domains (see, for example, [7]), making it central to the theory of Sobolev spaces and the study of partial differential equations.

The Hardy integral inequality is another pivotal result in analysis, establishing a relationship between a function and its integral mean (see, for example, [8]). The Hilbert integral inequality, a classical and influential result, provides an upper bound for a double integral involving a singular kernel function (see, for example, [8]). The Hardy-Hilbert integral inequality is a synthesis of these two results, establishing a double integral bound for non-negative functions with a symmetric kernel function. It naturally extends both the Hardy and Hilbert integral inequalities, unifying their analytical depth (see, for example, [8]).

Due to the significant implications of the Hardy-Hilbert integral inequality, many researchers have sought to expand and refine it in various ways. A comprehensive overview of these developments can be found in the survey [3] and the monograph [21]. Recent advances in the field include contributions exploring new forms, applications and generalisations within diverse analytical frameworks, as presented in [1, 2, 4–6, 9, 12, 13, 15–20].

In this article, we contribute to the theory of integral inequalities by introducing a new variation of the Hardy-Hilbert integral inequality involving a double integral of the form

$$\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy,$$

where f and g are two functions satisfying suitable integrability conditions, and μ and λ are real parameters subject to certain constraints. The novelty of our approach lies in the structure of the kernel function

$$K(x,y) = \frac{e^{-\lambda(x+y)}}{(x+y)^{\mu}},$$

which is notably inhomogeneous and exhibits a richer analytical behavior than the classical kernel functions considered in the previously cited works. Combining exponential decay with a singular power term, this kernel function provides a broader framework for studying weighted Hardy-Hilbert-type integral inequalities and their potential applications in functional and operator analysis. The constant factor obtained involves the gamma function. Building on this result, two other new integral inequalities are established.

The remainder of this article is organized as follows: In Section 2, we present some preliminary results and establish the notation used throughout the article. Section 3 develops the main theorem, together with its proof, examples, and other results. Finally, concluding remarks and potential directions for future research are given in Section 4.

2. Preliminaries and notations

For the sake of convenience, we set out several key notations in this section. Firstly, we recall the definitions of some well-known special functions. The gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

for any $x > 0$.

The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for any $x, y > 0$.

We also adopt standard notation from measure theory. Let $(X, \mathcal{F}, \nu)$ be a measure space and let $I \in \mathcal{F}$. For $p \in (1, \infty)$, we define

$$L^p(I) = \left\{ \nu\text{-measurable function } f : I \to \mathbb{R} : \int_I |f|^p d\nu < \infty \right\}.$$

Our focus will be on the case where X is the set of real numbers $\mathbb{R}$, $I = [0, \infty)$, and ν is the Lebesgue measure.

3. Results

The main result is highlighted in the next subsection, with two additional results presented in a subsequent one.

3.1. **Main Result.**

Our main result is established in the theorem below. We emphasize the originality of the kernel function and the constant factor that depends on the gamma function.

Theorem 3.1. *Let $p, q > 1$ be the Hölder conjugate exponents, i.e., $1/p + 1/q = 1$. Let $\mu \in (0, 1)$ and $\lambda > 0$. Then, for any Lebesgue measurable functions $f, g : [0, \infty) \to [0, \infty)$ with $f \in L^p([0, \infty))$ and $g \in L^q([0, \infty))$, we have*

$$\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy$$
$$\leq \omega(\lambda, \mu, p) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (g(y))^q dy \right)^{1/q},$$

where

$$\omega(\lambda, \mu, p) = p^{-1/p} q^{-1/q} \lambda^{\mu-1} \Gamma(1-\mu). \tag{3.1}$$

Proof. Firstly, using a standard property of the gamma function, the following integral relation comes from a natural change of variables:

$$\frac{1}{(x+y)^\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-t(x+y)} dt.$$

Hence, the explicit term of the integrand can be expressed as

$$\frac{e^{-\lambda(x+y)}}{(x+y)^\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-(t+\lambda)(x+y)} dt.$$

Let us make the change of variables $s = t + \lambda$. Then $t = s - \lambda$ and $t \in (0, \infty)$ corresponds to $s \in (\lambda, \infty)$. Thus

$$\frac{e^{-\lambda(x+y)}}{(x+y)^\mu} = \frac{1}{\Gamma(\mu)} \int_\lambda^\infty (s-\lambda)^{\mu-1} e^{-s(x+y)} ds.$$

Since all the terms in relation are non-negative, the Fubini-Tonelli integral theorem gives

$$\mathcal{J} := \int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy$$
$$= \frac{1}{\Gamma(\mu)} \int_\lambda^\infty (s-\lambda)^{\mu-1}\mathcal{I}(s)ds, \tag{3.2}$$

where

$$\mathcal{I}(s) := \left(\int_0^\infty f(x)e^{-sx}dx\right)\left(\int_0^\infty g(y)e^{-sy}dy\right).$$

Let us examine each of these integrals. By the Hölder integral inequality, we derive

$$\int_0^\infty f(x)e^{-sx}dx \leq \left(\int_0^\infty (f(x))^p dx\right)^{1/p}\left(\int_0^\infty e^{-sqx}dx\right)^{1/q}$$
$$= \left(\int_0^\infty (f(x))^p dx\right)^{1/p}(sq)^{-1/q}.$$

Similarly, we find that

$$\int_0^\infty g(y)e^{-sy}dy \leq \left(\int_0^\infty (g(y))^q dy\right)^{1/q}(sp)^{-1/p}.$$

Hence, using the equality $1/p + 1/q = 1$, we have

$$\mathcal{I}(s) \leq \left(\int_0^\infty (f(x))^p dx\right)^{1/p}\left(\int_0^\infty (g(y))^q dy\right)^{1/q} p^{-1/p}q^{-1/q}s^{-1}. \tag{3.3}$$

Combining Equations (3.2) and (3.3), we get

$$\mathcal{J} \leq \frac{p^{-1/p}q^{-1/q}}{\Gamma(\mu)}\left(\int_0^\infty (f(x))^p dx\right)^{1/p}\left(\int_0^\infty (g(y))^q dy\right)^{1/q}\int_\lambda^\infty (s-\lambda)^{\mu-1}s^{-1}ds. \tag{3.4}$$

Let us express the last integral term of this upper bound. We consider the change of variables $s = \lambda/(1-t)$, i.e., $t = 1-\lambda/s$. Then $s \in (\lambda, \infty)$ corresponds to $t \in (0,1)$. A calculation involving the beta function gives

$$\int_\lambda^\infty (s-\lambda)^{\mu-1}s^{-1}ds = \lambda^{\mu-1}\int_0^1 t^{\mu-1}(1-t)^{-\mu}dt = \lambda^{\mu-1}B(\mu, 1-\mu).$$

Using the formula $B(\mu, 1-\mu) = \Gamma(\mu)\Gamma(1-\mu)$, we obtain

$$\int_{\lambda}^{\infty} (s-\lambda)^{\mu-1} s^{-1} ds = \lambda^{\mu-1}\Gamma(\mu)\Gamma(1-\mu). \tag{3.5}$$

Combining Equations (3.4) and (3.5), and simplifying $\Gamma(\mu)$, we have

$$\mathcal{J} \leq p^{-1/p} q^{-1/q} \lambda^{\mu-1} \Gamma(1-\mu) \left(\int_0^{\infty} (f(x))^p dx \right)^{1/p} \left(\int_0^{\infty} (g(y))^q dy \right)^{1/q}.$$

By recognizing the definition of $\omega(\lambda, \mu, p)$, this completes the proof. □

Example 1. *For the special case* $\mu = 1/2$, *using* $\Gamma(1/2) = \sqrt{\pi}$, *we have* $\omega(\lambda, \mu, p) = p^{-1/p} q^{-1/q} \sqrt{\pi/\lambda}$ *and Theorem 3.1 gives the following elegant inequality:*

$$\int_0^{\infty} \int_0^{\infty} \frac{e^{-\lambda(x+y)}}{\sqrt{x+y}} f(x) g(y) dx dy$$
$$\leq p^{-1/p} q^{-1/q} \sqrt{\frac{\pi}{\lambda}} \left(\int_0^{\infty} (f(x))^p dx \right)^{1/p} \left(\int_0^{\infty} (g(y))^q dy \right)^{1/q}.$$

To the best of the knowledge of the author, the integral inequality stated in Theorem 3.1 is new. The constant obtained in this theorem is believed to be optimal because the proof relies on a minimal sequence of inequalities. However, we do not provide rigorous verification of this.

3.2. **Additional Results.** To demonstrate the flexibility and applicability of Theorem 3.1, we present below another integral inequality whose proof is built upon it.

Theorem 3.2. *Let* $p, q > 1$ *be the Hölder conjugate exponents, i.e.,* $1/p + 1/q = 1$. *Let* $\mu \in (0, 1)$ *and* $\lambda > 0$. *Then, for any Lebesgue measurable function* $f : [0, \infty) \to [0, \infty)$ *with* $f \in L^p([0, \infty))$, *we have*

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{e^{-\lambda(x+y)}}{(x+y)^{\mu}} f(x) dx \right)^p dy \leq \omega(\lambda, \mu, p)^p \int_0^{\infty} (f(x))^p dx,$$

where $\omega(\lambda, \mu, p)$ *is given by Equation* (3.1).

Proof. Let us set

$$\mathcal{K} := \int_0^{\infty} \left(\int_0^{\infty} \frac{e^{-\lambda(x+y)}}{(x+y)^{\mu}} f(x) dx \right)^p dy.$$

We can write

$$\mathcal{K} = \int_0^\infty \left(\int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)dx \right)^{p-1} \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g_\circ(y)dxdy, \tag{3.6}$$

where

$$g_\circ(y) = \left(\int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)dx \right)^{p-1}.$$

Applying Theorem 3.1 to the functions f and $g_\circ$, we get

$$\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g_\circ(y)dxdy$$

$$\leq \omega(\lambda, \mu, p) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (g_\circ(y))^q dy \right)^{1/q}. \tag{3.7}$$

Let us examine the last integral of this upper bound. Using the identity $q = p/(p-1)$, we have

$$\int_0^\infty (g_\circ(y))^q dy = \int_0^\infty \left(\int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)dx \right)^{q(p-1)} dy$$

$$= \int_0^\infty \left(\int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)dx \right)^p dy = \mathcal{K}. \tag{3.8}$$

Combining Equations (3.6), (3.7) and (3.8), we obtain

$$\mathcal{K} \leq \omega(\lambda, \mu, p) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \mathcal{K}^{1/q}.$$

Using $1 - 1/q = 1/p$ and raising to the exponent p, we get

$$\mathcal{K} \leq \omega(\lambda, \mu, p)^p \int_0^\infty (f(x))^p dx,$$

which is the desired inequality, completing the proof. □

Example 2. *For the special case $\mu = 1/2$, using $\Gamma(1/2) = \sqrt{\pi}$, Theorem 3.2 gives the elegant inequality*

$$\int_0^\infty \left(\int_0^\infty \frac{e^{-\lambda(x+y)}}{\sqrt{x+y}} f(x)dx \right)^p dy \leq p^{-1/p} q^{-1/q} \sqrt{\frac{\pi}{\lambda}} \int_0^\infty (f(x))^p dx.$$

Another integral inequality that builds on Theorem 3.1 is provided below. It is based on an integral approach. We emphasize the originality of both the kernel function and the constant factor.

Theorem 3.3. *Let* $p, q > 1$ *be the Hölder conjugate exponents, i.e.,* $1/p + 1/q = 1$. *Let* $\mu \in (0,1)$ *and* $\theta > 0$. *Then, for any Lebesgue measurable functions* $f, g : [0,\infty) \to [0,\infty)$ *with* $f \in L^p([0,\infty))$ *and* $g \in L^q([0,\infty))$, *we have*

$$\int_0^\infty \int_0^\infty \frac{1 - e^{-\theta(x+y)}}{(x+y)^{\mu+1}} f(x)g(y)dxdy \\ \leq \xi(\theta,\mu,p) \left(\int_0^\infty (f(x))^p dx\right)^{1/p} \left(\int_0^\infty (g(y))^q dy\right)^{1/q},$$

where

$$\xi(\theta,\mu,p) = p^{-1/p} q^{-1/q} \mu^{-1} \theta^\mu \Gamma(1-\mu). \tag{3.9}$$

Proof. It follows from Theorem 3.1 that

$$\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy \\ \leq \omega(\lambda,\mu,p) \left(\int_0^\infty (f(x))^p dx\right)^{1/p} \left(\int_0^\infty (g(y))^q dy\right)^{1/q},$$

where $\omega(\lambda,\mu,p)$ is given by Equation (3.1). This is valid for $\mu \in (0,1)$ and $\lambda > 0$.

Considering λ as a variable and integrating with respect to $\lambda \in (0,\theta)$, we get

$$\int_0^\theta \left(\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy\right) d\lambda \\ \leq \left(\int_0^\theta \omega(\lambda,\mu,p)d\lambda\right) \left(\int_0^\infty (f(x))^p dx\right)^{1/p} \left(\int_0^\infty (g(y))^q dy\right)^{1/q}. \tag{3.10}$$

Simple power primitives give

$$\int_0^\theta \omega(\lambda,\mu,p)d\lambda = p^{-1/p} q^{-1/q} \left(\int_0^\theta \lambda^{\mu-1} d\lambda\right) \Gamma(1-\mu) \\ = p^{-1/p} q^{-1/q} \mu^{-1} \theta^\mu \Gamma(1-\mu) = \xi(\theta,\mu,p). \tag{3.11}$$

Furthermore, applying the Fubini-Tonelli integral theorem, we have

$$\begin{aligned}
&\int_0^\theta \left(\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x+y)}}{(x+y)^\mu} f(x)g(y)dxdy \right) d\lambda \\
&= \int_0^\infty \int_0^\infty \left(\int_0^\theta e^{-\lambda(x+y)} d\lambda \right) \frac{1}{(x+y)^\mu} f(x)g(y)dxdy \\
&= \int_0^\infty \int_0^\infty \left(\frac{1-e^{-\theta(x+y)}}{x+y} \right) \frac{1}{(x+y)^\mu} f(x)g(y)dxdy \\
&= \int_0^\infty \int_0^\infty \frac{1-e^{-\theta(x+y)}}{(x+y)^{\mu+1}} f(x)g(y)dxdy. \qquad (3.12)
\end{aligned}$$

Combining Equations (3.10), (3.11) and (3.12), we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{1-e^{-\theta(x+y)}}{(x+y)^{\mu+1}} f(x)g(y)dxdy \\
&\leq \xi(\theta,\mu,p) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (g(y))^q dy \right)^{1/q},
\end{aligned}$$

completing the proof. □

Example 3. *For the special case $\mu = 1/2$, using $\Gamma(1/2) = \sqrt{\pi}$, we have $\xi(\theta,\mu,p) = 2p^{-1/p}q^{-1/q}\sqrt{\theta\pi}$ and Theorem 3.3 gives the following elegant inequality:*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{1-e^{-\theta(x+y)}}{(x+y)^{3/2}} f(x)g(y)dxdy \\
&\leq 2p^{-1/p}q^{-1/q}\sqrt{\theta\pi} \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (g(y))^q dy \right)^{1/q}.
\end{aligned}$$

4. CONCLUSION

In conclusion, we have established a new Hardy-Hilbert-type integral inequality featuring a kernel function that combines exponential decay with a singular power term. The obtained results highlight the versatility of this approach and its potential to generate further integral inequalities of similar nature. Future research may focus on extending these results to multidimensional settings, for instance, by considering the triple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x)g(y)h(z)}{(x+y+z)^\mu} e^{-\lambda(x+y+z)} dxdydz,$$

exploring optimality conditions for the constant factor, and applying the inequalities to related problems in analysis and mathematical physics.

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