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RATLIFF-RUSH FILTRATION INDUCED BY A GOOD FILTRATION

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ABSTRACT. In this note, we introduce and study the notion of Ratliff-Rush filtration associated with an good filtration which generalize the notion of Ratliff-Rush closure of an ideal introduce by L.J. Ratliff and D.Rush. We establish a semi prime operation in the class of good filtrations which is a refinement of the pruferian closure of filtration, we generalize a theorem of Samuel on power of ideals.

1. INTRODUCTION

Let A be a commutative ring and I be an ideal of A. The Ratliff-Rush ideal associated to I is defined in [8] by:

$$\tilde{I} = \bigcup_{i\geq 0}(I^{i+1} : I^i).$$

This ideal has been studied in [2–4, 9]. Some nice proprieties has been stablished, for instance if I is regular then:

$$I \subset \tilde{I} \subset \bar{I} \subset \sqrt{I},$$

where $\bar{I}$ is the integral closure of I. For all large n,

$$\widetilde{I^n} = I^n.$$

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The notion of Ratliff-Rush filtration associated to an ideal has been defined in [10], [12] by the sequence:

$$\{\widetilde{I^n} = \underset{i\geq 0}{\cup}(I^{n+i} : I^i)\}_{n\in\mathbb{N}}.$$

In [11], the autors introduce the notion of Ratliff-Rush filtration associated to an I-filtration by the family:

$$\{\widetilde{I_n} = \underset{i\geq 0}{\cup}(I_{n+i} : I^i)\}_{n\in\mathbb{N}}.$$

In this paper we introduce and study the notion of Ratliff-Rush filtration associated to an good filtration.

In section 1, we recall a few preliminaries concerning the definitions and properties of filtrations.

Section 2 is devoted to the study of Ratliff-Rush filtration associated to an good filtration. Let $f = (I_n)_{n\in\mathbb{N}}$ be good filtration of the ring A, the Ratliff-Rush filtration associated to f is the sequence

$$\widehat{f} = \{\widehat{I_n} = \underset{i\geq 0}{\cup}(I_{n+i} : I_i)\}_{n\in\mathbb{N}}.$$

We show in Lemma 3.3 that there exists an integer N, such that for all n,

$$\widehat{I_n} = \cup_{i=1}^{N}(I_{n+i} : I_i) = I_{n+N} : I_N.$$

In Lemma 3.2 we prove that for an f-superficial and regular element x and for all $n >> 0$,

$$\widehat{I_n} = I_{n+N} : xA = I_n.$$

In the section 3 we show that if f is regular then:

$$f \leq \widehat{f} \leq P(f),$$

where $P(f)$ is the prüferian closure of f, and the operation

$$f \longmapsto \widehat{f}$$

is a prime operation on the set of good filtrations see Proposition 3.2.

We end with an example 3.1 that calculates $\widehat{f}$.

2. Preliminary Notes

We start by giving the basic definitions. The rings considered are commutative and unitary.

Definition 2.1. *A filtration on a ring A is a family $f = (I_n)_{n\in\mathbb{N}}$ of ideals of A such that:*

$$I_0 = A,\ I_{n+1} \subseteq I_n,\ I_n I_m \subseteq I_{n+m},\ \forall n, m \in \mathbb{N}.$$

The set of filtrations of A is denoted by $\mathbb{F}(A)$.

If I is an ideal of A then the family $f_I = (I^n)_{n\in\mathbb{N}}$ is a filtration of A called the I-adic filtration.

Definition 2.2. *Let M be a module of A, a filtration of M is a family $\Phi = (M_n)_{n\in\mathbb{N}}$ of sub-modules of M such that:*

$$M_0 = M, M_{n+1} \subseteq M_n, \forall n \in \mathbb{N}.$$

Let $f = (I_n)_{n\in\mathbb{N}} \in \mathbb{F}(A)$, a filtration $\Phi = (M_n)_{n\in\mathbb{N}}$ of M is said to be compatible with f if $I_n M_p \subseteq M_{n+p}$ for all $n, p \in \mathbb{N}$.

Definition 2.3. *Let J be an ideal of A, a filtration $f = (I_n)_{n\in\mathbb{N}}$ of A is said to be $J - good$ if $J \subset I_1$ and if there is an integer $r \geq 1$ such that*

$$I_{n+1} = JI_n\ \forall n \geq r.$$

Definition 2.4. *A filtration $f = (I_n)_{n\in\mathbb{N}}$ of A is said to be noetherian if the Rees ring $\mathrm{R}(A, f) = \underset{n\geq 0}{\oplus} I_n X^n$ is noetherian.*

Remark 2.1. *If A is noetherian, then the filtration $f = (I_n)_{n\in\mathbb{N}}$ of A is noetherian if, only if, there is an integer $k \geq 1$ such that: $I_{n+k} = I_n I_k$ for all $n \geq k$ [1].*

Therefore any good filtration is a nooetherian filtration in a noetherian ring.

Definition 2.5. *If $f = (I_n)_{n\in\mathbb{N}}$ of A is noetherian then we call the order of f the smallest integer $k \geq 1$ such that $I_{n+k} = I_n I_k$.*

Definition 2.6. *Let $f = (I_n)_{n\in\mathbb{N}}$ and $g = (J_n)_{n\in\mathbb{N}}$ be two filtrations of A such that $f \leq g$ (i.e $I_n \subseteq J_n$), f is said to be a reduction of g if there are the integers $k \geq 1$ and $n_0 \in \mathbb{N}$ such that*

$$\forall n \geq n_0, J_{n+k} = I_n J_k.$$

Definition 2.7. *Let A be a noetherian ring and $f = (I_n)_{n\in\mathbb{N}}$ be a noetherian filtration of A. An element $x \in A$ is said to be f-superficial of degree $k \in \mathbb{N}^*$ if $x \in I_k$ and there is an integer $c \in \mathbb{N}$ such that*

$$(I_{n+k} : x) \cap I_c = I_n \ \forall n \geq c.$$

Remark 2.2. *If I_1 contains an regular element, then there exist an f-superficial element which is regular, Corollary 8.5.9 of [6].*

Definition 2.8. *A filtration $f = (I_n)_{n\mathbb{N}}$ on a ring A is said to be regular if I_1 contains a regular element.*

Definition 2.9. *An element $x \in A$ is said to be integral over the filtration $f = (I_n)_{n\mathbb{N}}$ if x satisfies an equation: $x^m + a_1x^{m-1} + ... + a_jx^{m-j} + ... + a_m = 0$ where $a_j \in I_j$ for all j. Let $k \in \mathbb{N}$, we put $P_k(f) = \{x \in A,$ integral on $f^{(k)}\}$, where $f^{(k)} = (I_{nk})_{n\mathbb{N}}$. The sequence $P(f) = (P_k(f))_{k\in\mathbb{N}}$ is a filtration of A, called the prüferian closure of f.*

3. Ratliff-Rush filtration

Lemma 3.1. *Let A be a ring and $f = (I_n)_{n\in\mathbb{N}}$ be a good filtration of A. For any integer k, for all $n \geq 1$, we have:*

$$I_{n+k} : I_n \subseteq I_{m+k} : I_m, \quad \textit{for all} m \gg 0.$$

Proof. Let $x \in I_{n+k} : I_n$, then $xI_n \subset I_{n+k}$. For $m \geq n+r$ we have: $xI_m = xI_{n+m-n} \subseteq xI_nI_{m-n} \subseteq I_{n+k}I_{m-n} \subseteq I_{m+k}$. Then: $I_{n+k} : I_n \subseteq I_{m+k} : I_m$. □

Proposition 3.1. *Let A be a ring and $f = (I_n)_{n\in\mathbb{N}}$ be a good filtration of A. For all integer k we put $\widehat{I_k} = \underset{n\geq 1}{\cup}(I_{n+k} : I_n)$ and $\widehat{f} = (\widehat{I_k})_{k\in\mathbb{N}}$, then $\widehat{f}$ is a filtration of A.*

Proof. Let $\widehat{I_k}$ is an ideal of A for all integer k in application of lemma 3.1, and let $x \in \widehat{I_{k+1}}$, there exist $n \in \mathbb{N}$ such that $xI_n \subseteq I_{n+k+1}$. We have $xI_n \subseteq I_{n+k+1} \subseteq I_{n+k}$ then $x \in \widehat{I_k}$ so $\widehat{I_{k+1}} \subseteq \widehat{I_k}$.

Let $x \in \widehat{I_p}$, and $y \in \widehat{I_q}$, then there exist $n \in \mathbb{N}$ such that $x \in I_{n+p} : I_n$, and $m \in \mathbb{N}$ so $y \in I_{m+q} : I_m$. For n, m large:

$$xyI_{n+m} = xyI_nI_m = xI_nyI_m \subseteq I_{n+p}I_{m+q} \subseteq I_{n+m+p+q}$$

we have $xy \in \widehat{I_{p+q}}$, then $\widehat{I_p}\widehat{I_q} \subseteq \widehat{I_{p+q}}$. □

Lemma 3.2. *Let A be a noetherian ring, $f = (I_n)_{n\in\mathbb{N}}$ a good filtration, x a regular and f-superficial element of degree k. Then for all large n we have:*

$$I_{n+k} : xA = I_n.$$

Proof. First $I_n \subset I_{n+k} : xA$ since $x \in I_k$. Now, let $y \in I_{n+k} : xA$ with n large then: $yx \in I_{n+k} \cap xA = I_n(I_k \cap xA)$ consequently $yx = \lambda x$ where $\lambda \in I_n$. Since x is regular,so $y \in I_n$, therefore $I_{n+k} : xA \subseteq I_n$. □

Lemma 3.3. *Let A be a noetherian ring, $f = (I_n)_{n\in\mathbb{N}}$ a good filtration of A, then $\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$:*

$$\widehat{I_n} = \underset{i\geq 0}{\cup}(I_{n+i} : I_i) = I_{n+N} : I_N = I_{n+i} : I_i \,, \forall i \geq N.$$

Proof. There exists $r \in \mathbb{N}$, such that for all $i \geq r$, $I_{n+i} : I_i \subset I_{n+i+1} : I_{i+1}$. Then, there exists $N \in \mathbb{N}$: $I_{n+i} : I_i = I_{n+N} : I_N$. For all $i \leq r-1$, $I_{n+i} : I_i \subset I_{n+N} : I_N$. Therefore

$$\widehat{I_n} = I_{n+N} : I_N = I_{n+i} : I_i \,, \quad \text{for all } i \geq N.$$

□

Theorem 3.1. *Let A be a noetherian ring, $f = (I_n)_{n\in\mathbb{N}}$ a good filtration of A, with I_1 contain an regular and $f-$superficial element. Then, for $n >> 0$, $\widehat{I_n} = \underset{i\geq 0}{\cup}(I_{n+i} : I_i) = I_n$.*

Proof. For all $n, I_n \subset \underset{i\geq 0}{\cup}(I_{n+i} : I_i)$. Using lemma 3.2 and lemma 3.3, there exists $x \in A$ and $N \in \mathbb{N}$, such that: $I_{n+N} : xA = I_n$, for all $n \geq N$, $\underset{i\geq 0}{\cup}(I_{n+i} : I_i) = I_{n+N} : I_N \subset I_{n+N} : xA = I_n$ for all $n \geq N$. □

Corollary 3.1. *Let A be a noetherian ring, $f = (I_n)_{n\in\mathbb{N}}$ a good filtration of A, with I_1 contain an regular and $f-$superficial element. Then: $\widehat{f}$ is a good filtration.*

Proof. According to the previous theorem, for all $n >> 0, \widehat{I_n} = I_n$. As f is a good filtration then $\widehat{f}$ is also □

Proposition 3.2. *Let A be a noetherian ring, I an ideal of A, then the map $f \mapsto \widehat{f}$ is a semi prime operation in the class of I- good filtration containing an regular superficial element, which mean that:*

i) $f \leq \widehat{f}$

ii) $\widehat{\widehat{f}} = \widehat{f}$
iii) $f \leq g \Rightarrow \widehat{f} \leq \widehat{g}$
iv) $\widehat{f}\widehat{g} \leq \widehat{fg}$

Proof.
i) Let $n \in \mathbb{N}$, and $i \in \mathbb{N}$. Then $I_n \subseteq I_{n+i} : I_i$, so $I_n \subseteq \widehat{I_n}$ and $f \leq \widehat{f}$.

ii) Let $n \in \mathbb{N}$. Using lemma 3.2 and thoerem 3.1, there exists N such that $\widehat{I_n} = I_{N+n} : I_N$, $\widehat{\widehat{I_n}} = (\widehat{J_{N+n}} : \widehat{J_N})$ where $J_n = \widehat{I_n}$. For N big enough, we have: $\widehat{J_{n+N}} = J_{n+N}$ and $\widehat{J_N} = J_N$, so $\widehat{\widehat{I_n}} = (J_{n+N} : J_N) = J_n = \widehat{I_n}$.

iii) Let $f = (I_n)$ and $g = (J_n)$ are I-good, such that for all n $I_n \subset J_n$. According to theorem 3.1, there exist $N \geq 1$ such that: $\widehat{I_n} = I_{N+n} : I_N$ and $\widehat{J_n} = J_{N+n} : J_N$.

Let $x \in \widehat{I_n}$. Then $xI_N \subset I_{N+n}$. Since g is $I-$ good there exist an integer n_0 such that $J_{n+n_0} = I^n J_{n_0}$ for all n. For $N >> 0$, we have $xJ_N = xI^{N-n_0}J_{n_0} \subset xI_{N+n_0}J_{n_0} \subset I_{n+N+n_0}J_{n_0}J_{n+N+n_0}J_{n_0} \subset J_{n+N}$. Then $\widehat{I_n} \subset \widehat{J_n}$, therefore $\widehat{f} \leq \widehat{g}$.

iv) There exist $N \geq 1$ such that: $\widehat{I_n} = I_{N+n} : I_N$ and $\widehat{J_n} = J_{N+n} : J_N$. Let

$$x \in \widehat{I_n} = I_{n+N} : I_N$$

and

$$y \in \widehat{J_n} = J_{n+N} : J_N.$$

Then

$$xyI_N J_N = xI_N yJ_N \subseteq I_{n+N}J_{n+N},$$

and so

$$xy \in I_{n+N}J_{n+N} : I_N J_N \ = \widehat{I_n J_n}$$

and

$$\widehat{I_n}\widehat{J_n} \subseteq \widehat{I_n J_n}.$$

Therefore

$$\widehat{f}\widehat{g} \leq \widehat{fg}.$$

□

Proposition 3.3. *Let A be a noetherian ring, $f = (I_n)_{n\in\mathbb{N}}$ a good filtration of A, with I_1 contain an regular and $f-$superficial element then $f \leq \widehat{f} \leq P(f)$ where $P(f)$ is the pruferian closer of f.*

Proof. According to theorem 3.1, there exists an integer N such that: $\widehat{I_{n+N}} = I_{n+N} = I_n I_n = I_n \widehat{I_N}$ which proves that f is a reduction of $\widehat{f}$. Then $\widehat{f} \leq P(f)$ (see [[5], (4.6)]). □

Example 3.1. *Let k be a field, $A = k[X, Y]$,*

$$I = (X^2, Y^2), \quad J = (X^2, Y^2, XY),$$

and I and J are two ideals of the ring A. We have

$$I \subset J$$

and

$$IJ = (X^4, X^3Y, X^2Y^2, XY^3, Y^4) = J^2.$$

So, I is a reduction of J. We put

$$I_0 = A, I_1 = I,$$

and for all $n \geq 2$,

$$I_n = IJ^{n-1}.$$

We have

$$I_{n+1} \subset I_n$$

and for all $n \in \mathbb{N}$ and $p \geq 2$,

$$I_n I_p = IJ^{n-1} IJ^{p-1} = I^2 J^{n+p-2} \subset IJ^{n+p-1} \subset I_{n+p}.$$

For all $n \geq 2$,

$$II_n = I_{n+1}.$$

So $f = (I_n)_{n \in \mathbb{N}}$ is a good filtration on the ring A.

Let's show that, for all $n \in \mathbb{N}$, $k \in \mathbb{N}$, $k \geq 2$,

$$J^{n+k} : J^k = J^n.$$

Put $L = (X, Y)$. Then $L^2 = J$. Let $z \in J^{n+k} : J^k$ be an homogeneous element. Then

$$zL^{2k} \subset J^{n+k} = L^{2n+2k},$$

thus $d(z) = 2n + 2k - 2k = 2n$. So $z \in L^{2n} = J^n$. Therefore,

$$J^{n+k} : J^k \subset J^n,$$

and the equality holds since the inverse inclusion is trivial.

We have

$$\cup_{k\geq 2}(I_{n+k} : I_k) = \cup_{k\geq 2}(J^{n+k} : J^k) = J^n.$$

So, for all n we have

$$\cup_{k\geq 2}(I_{n+k} : I_k) = \cup_{k\geq 2}(J^{n+k} : J^k) = J^n.$$

Then

$$\begin{aligned}\widehat{I_n} &= \cup_{k\geq 1}(I_{n+k} : I_k)\\ &= (I_{n+1} : I_1) \cup J^n\\ &= (I_{n+1} : I_1)\\ &= (J^{n+1} : I_1)\\ &= J^n.\end{aligned}$$

So,

$$\widehat{f} = (J^n)_{n\in\mathbb{N}}$$

is the J-adic filtration.

Conflict of interest

Authors declare that there is no conflict of interest related to this paper.

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