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ON A NEW MULTI-DIMENSIONAL TWO-PARAMETER HARDY-HILBERT-TYPE INTEGRAL INEQUALITY

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ABSTRACT. This article investigates a new three-dimensional Hardy-Hilbert-type integral inequality involving an exponential-ratio kernel function that is governed by two parameters. An explicit upper bound is established, with the corresponding constant factor expressed in terms of the gamma function. This result is then extended to a multi-dimensional setting to demonstrate the generality and analytical flexibility of the proposed approach.

1. INTRODUCTION

The Hardy-Hilbert integral inequality is a classical and influential result in analysis, thoroughly studied in [8]. Over the years, numerous researchers have sought to extend, refine, and generalize this inequality in various directions. A comprehensive overview of these developments is provided in the survey [3] and the book [18]. Recent advances include the introduction of new forms, applications to diverse analytical contexts, and broader generalizations, as explored

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in [1, 2, 4–6, 9–17]. These contributions collectively highlight the continuing relevance and versatility of Hardy-Hilbert-type integral inequalities in modern analysis.

A new variant of the Hardy-Hilbert integral inequality was introduced in [7]. It features a kernel function of the exponential-ratio type that depends on two parameters and exhibits a decreasing behavior. One of the research directions highlighted in [7] concerns extending this inequality to higher dimensions. In particular, it is proposed investigating inequalities based on the following three-dimensional integral analogue:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{e^{-\gamma(x+y+z)}}{(x+y+z)^{\lambda}}f(x)g(y)h(z)dxdydz,$$

where $\gamma > 0$, $\lambda \in (0, 2)$, and f, g, h are non-negative measurable functions that satisfy suitable integrability conditions. In this article, we build upon this line of research by deriving an explicit upper bound for this triple integral. The resulting constant factor involves the gamma function. Furthermore, we extend the analysis to a multi-dimensional setting to demonstrate the generality and flexibility of the approach. We provide detailed proofs of the main results, emphasizing both the analytical technique and the structural properties of the kernel function.

The remainder of the article is organized as follows: Section 2 presents the preliminary notions and results required for our analysis. Section 3 is devoted to the establishment of the main three-dimensional Hardy-Hilbert-type integral inequality. The corresponding multi-dimensional analogue is investigated in Section 4. Finally, concluding remarks are provided in Section 5.

2. Preliminaries

2.1. **Gamma and beta functions, and L^p space.** For the sake of clarity and convenience, we begin by recalling some classical special functions. First, the gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

for any $x > 0$.

The beta function is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for any $x, y > 0$.

Note that, for any positive integer m, we have $\Gamma(m) = (m-1)!$.

We also adopt standard notation from measure theory. Let $(X, \mathcal{F}, \nu)$ be a measure space and let $I \in \mathcal{F}$. For $p \in (1, \infty)$, we define

$$L^p(I)=\left\{\nu\text{-measurable function } f : I \to \mathbb{R} : \|f\|_p^p = \int_I |f|^p d\nu < \infty\right\}.$$

Our focus will be on the case where X is the set of real numbers $\mathbb{R}$, $I = [0, \infty)$, and ν is the Lebesgue measure.

2.2. **Basis theorem.** For completeness and ease of reference, we recall the statement of [7, Theorem 1] below.

Theorem 2.1. *[7, Theorem 1] Let* $p, q > 1$ *satisfy* $1/p + 1/q = 1$*, and* f, g *be non-negative measurable functions such that* $f \in L^p([0,\infty))$ *and* $g \in L^q([0,\infty))$*. Let* $\lambda \in (0, 1)$ *and* $\gamma > 0$*. Then the following inequality holds:*

$$\int_0^\infty\int_0^\infty \frac{e^{-\gamma(x+y)}}{(x+y)^\lambda} f(x)g(y)dxdy \le \Xi\|f\|_p\|g\|_q,$$

where

$$\Xi = p^{-1/p}q^{-1/q}\gamma^{\lambda-1}\Gamma(1-\lambda).$$

The main difference between this inequality and the classical Hardy-Hilbert integral inequality is that it involves an exponential-ratio kernel function depending on two parameters, rather than the simple ratio-sum kernel function $1/(x+y)$. This modification introduces a decaying exponential weight, which enhances convergence and allows for a richer class of integrable functions. We emphasize that the gamma function is fundamental in the formulation of the constant factor. Moreover, the parameter value $\lambda = 1$ is omitted from the analysis, as it leads to divergence in the gamma function term.

3. Three-dimensional result

Our main three-dimensional result is stated in the theorem below. We highlight the originality of the kernel function and the explicit form of the constant factor, which involves the gamma function.

Theorem 3.1. *Let $p, q, r > 1$ satisfy $1/p + 1/q + 1/r = 1$, and f, g, h be non-negative measurable functions such that $f \in L^p([0,\infty))$, $g \in L^q([0,\infty))$, and $h \in L^r([0,\infty))$. Let $\lambda \in (0,2)$ and $\gamma > 0$. Then the following inequality holds:*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\gamma(x+y+z)}}{(x+y+z)^\lambda} f(x)g(y)h(z)dxdydz \leq \Upsilon \|f\|_p \|g\|_q \|h\|_r,$$

where

$$\Upsilon = \left(\frac{p}{p-1}\right)^{1/p-1} \left(\frac{q}{q-1}\right)^{1/q-1} \left(\frac{r}{r-1}\right)^{1/r-1} \gamma^{\lambda-2}\Gamma(2-\lambda).$$

Proof. The change of variables $u = t(x+y+z)$ and the use of the gamma function yield

$$\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t(x+y+z)} dt = \frac{1}{(x+y+z)^\lambda} \frac{1}{\Gamma(\lambda)} \int_0^\infty u^{\lambda-1} e^{-u} du = \frac{1}{(x+y+z)^\lambda}.$$

Hence, by the Fubini-Tonelli integral theorem, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\gamma(x+y+z)}}{(x+y+z)^\lambda} f(x)g(y)h(z)dxdydz \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \int_0^\infty \int_0^\infty \int_0^\infty f(x)g(y)h(z)e^{-(\gamma+t)(x+y+z)} dxdydzdt. \end{aligned} \tag{3.1}$$

Let us focus on the main triple integral with respect to x, y and z. By the separability of the involved functions, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \int_0^\infty f(x)g(y)h(z)e^{-(\gamma+t)(x+y+z)} dxdydz \\ &= \int_0^\infty f(x)e^{-(\gamma+t)x} dx \int_0^\infty g(y)e^{-(\gamma+t)y} dy \int_0^\infty h(z)e^{-(\gamma+t)z} dz. \end{aligned} \tag{3.2}$$

Let us now bound each of these simple integrals. Let $p_*, q_*, r_* > 1$ satisfy

$$\frac{1}{p} + \frac{1}{p_*} = 1, \quad \frac{1}{q} + \frac{1}{q_*} = 1, \quad \frac{1}{r} + \frac{1}{r_*} = 1,$$

so that

$$p_* = \frac{p}{p-1}, \quad q_* = \frac{q}{q-1}, \quad r_* = \frac{r}{r-1}.$$

Applying the Hölder integral inequality with the parameters p and p_*, we get

$$\int_0^\infty f(x)e^{-(\gamma+t)x}dx \leq \|f\|_p \left(\int_0^\infty e^{-p_*(\gamma+t)x}dx\right)^{1/p_*} = \|f\|_p[p_*(\gamma+t)]^{-1/p_*}$$
$$= \|f\|_p p_*^{-1/p_*}(\gamma+t)^{-1/p_*}.$$

Similarly, we have

$$\int_0^\infty g(y)e^{-(\gamma+t)y}dy \leq \|g\|_q \left(\int_0^\infty e^{-q_*(\gamma+t)y}dy\right)^{1/q_*} = \|g\|_q[q_*(\gamma+t)]^{-1/q_*}$$
$$= \|g\|_q q_*^{-1/q_*}(\gamma+t)^{-1/q_*}$$

and

$$\int_0^\infty h(z)e^{-(\gamma+t)z}dz \leq \|h\|_r \left(\int_0^\infty e^{-r_*(\gamma+t)z}dz\right)^{1/r_*} = \|h\|_r[r_*(\gamma+t)]^{-1/r_*}$$
$$= \|h\|_r r_*^{-1/r_*}(\gamma+t)^{-1/r_*}.$$

Multiplying these bounds and using

$$\frac{1}{p_*} + \frac{1}{q_*} + \frac{1}{r_*} = 1 - \frac{1}{p} + 1 - \frac{1}{q} + 1 - \frac{1}{r} = 3 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = 3 - 1 = 2,$$

we obtain

$$\int_0^\infty f(x)e^{-(\gamma+t)x}dx \int_0^\infty g(y)e^{-(\gamma+t)y}dy \int_0^\infty h(z)e^{-(\gamma+t)z}dz$$
$$\leq \|f\|_p\|g\|_q\|h\|_r p_*^{-1/p_*}q_*^{-1/q_*}r_*^{-1/r_*}(\gamma+t)^{-(1/p_*+1/q_*+1/r_*)}$$
$$= \|f\|_p\|g\|_q\|h\|_r p_*^{-1/p_*}q_*^{-1/q_*}r_*^{-1/r_*}(\gamma+t)^{-2}. \tag{3.3}$$

Combining Equations (3.1), (3.2) and (3.3), we get

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\gamma(x+y+z)}}{(x+y+z)^\lambda} f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}\|f\|_p\|g\|_q\|h\|_r p_*^{-1/p_*}q_*^{-1/q_*}r_*^{-1/r_*}(\gamma+t)^{-2}dt$$
$$= \left(\frac{1}{\Gamma(\lambda)} p_*^{-1/p_*}q_*^{-1/q_*}r_*^{-1/r_*} \int_0^\infty t^{\lambda-1}(\gamma+t)^{-2}dt\right) \|f\|_p\|g\|_q\|h\|_r. \tag{3.4}$$

Let us now focus on the simple integral with respect to t. Changing the variables $u = t/\gamma$ and using standard properties of the beta and gamma functions, we have

$$\int_0^{\infty} t^{\lambda-1}(\gamma+t)^{-2}dt = \gamma^{\lambda-2}\int_0^{\infty} u^{\lambda-1}(1+u)^{-2}du = \gamma^{\lambda-2}B(\lambda, 2-\lambda)$$

$$= \gamma^{\lambda-2}\frac{\Gamma(\lambda)\Gamma(2-\lambda)}{\Gamma(2)} = \gamma^{\lambda-2}\Gamma(\lambda)\Gamma(2-\lambda). \tag{3.5}$$

Combining Equations (3.4) and (3.5), simplifying $\Gamma(\lambda)$ and using the expressions of p_*, q_* and r_*, we obtain

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{e^{-\gamma(x+y+z)}}{(x+y+z)^{\lambda}} f(x)g(y)h(z)dxdydz$$

$$\leq \left(\frac{1}{\Gamma(\lambda)} p_*^{-1/p_*} q_*^{-1/q_*} r_*^{-1/r_*} \gamma^{\lambda-2}\Gamma(\lambda)\Gamma(2-\lambda)\right) \|f\|_p\|g\|_q\|h\|_r$$

$$= \left(\left(\frac{p}{p-1}\right)^{1/p-1}\left(\frac{q}{q-1}\right)^{1/q-1}\left(\frac{r}{r-1}\right)^{1/r-1}\gamma^{\lambda-2}\Gamma(2-\lambda)\right)\|f\|_p\|g\|_q\|h\|_r$$

$$= \Upsilon\|f\|_p\|g\|_q\|h\|_r,$$

which completes the proof. □

Therefore, Theorem 3.1 establishes a new three-dimensional Hardy-Hilbert-type integral inequality with an exponential-ratio kernel function. The constant factor Υ depends explicitly on the gamma function, the decay parameters λ and γ, and the exponents p, q, r, reflecting the balance between the singularity at the origin and the exponential decay at infinity.

In the simple case where $\lambda = 1 \in (0, 2)$, Theorem 3.1 reduces to

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{e^{-\gamma(x+y+z)}}{x+y+z} f(x)g(y)h(z)dxdydz \leq \Upsilon\|f\|_p\|g\|_q\|h\|_r,$$

where

$$\Upsilon = \left(\frac{p}{p-1}\right)^{1/p-1}\left(\frac{q}{q-1}\right)^{1/q-1}\left(\frac{r}{r-1}\right)^{1/r-1}\gamma^{-1}.$$

4. Multi-dimensional result

Our main multi-dimensional result is presented in the theorem below. As in Theorem 3.1, we highlight the originality of the kernel function and the explicit form of the constant factor, which involves the gamma function.

Theorem 4.1. *Let $n \in \mathbb{N}\backslash\{0,1\}$, $p_1, \ldots, p_n > 1$ satisfy $\sum_{i=1}^{n} 1/p_i = 1$ and $f_1, \ldots, f_n$ be non-negative measurable functions such that, for any $i = 1, \ldots, n$, $f_i \in L^{p_i}([0,\infty))$. Let $\lambda \in (0, n-1)$ and $\gamma > 0$. Then the following inequality holds:*

$$\int_0^\infty \ldots \int_0^\infty \frac{e^{-\gamma \sum_{i=1}^n x_i}}{\left(\sum_{i=1}^n x_i\right)^\lambda} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n \leq \Omega \|f_1\|_{p_1} \ldots \|f_n\|_{p_n},$$

where

$$\Omega = \left(\frac{p_1}{p_1 - 1}\right)^{1/p_1 - 1} \ldots \left(\frac{p_n}{p_n - 1}\right)^{1/p_n - 1} \gamma^{\lambda - (n-1)} \frac{1}{(n-2)!} \Gamma(n - 1 - \lambda).$$

Proof. The proof follows the same main lines to that of Theorem 3.1. The change of variables $u = t\left(\sum_{i=1}^n x_i\right)$ and the use of the gamma function yield

$$\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t\sum_{i=1}^n x_i} dt = \frac{1}{\left(\sum_{i=1}^n x_i\right)^\lambda} \frac{1}{\Gamma(\lambda)} \int_0^\infty u^{\lambda-1} e^{-u} du = \frac{1}{\left(\sum_{i=1}^n x_i\right)^\lambda}.$$

Hence, by the Fubini-Tonelli integral theorem, we have

$$\int_0^\infty \ldots \int_0^\infty \frac{e^{-\gamma \sum_{i=1}^n x_i}}{\left(\sum_{i=1}^n x_i\right)^\lambda} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \int_0^\infty \ldots \int_0^\infty f_1(x_1) \ldots f_n(x_n) e^{-(\gamma+t)\sum_{i=1}^n x_i} dx_1 \ldots dx_n dt. \tag{4.1}$$

Let us focus on the main multi-dimensional integral with respect to $x_1, \ldots, x_n$. By the separability of the involved functions, we have

$$\int_0^\infty \ldots \int_0^\infty f_1(x_1) \ldots f_n(x_n) e^{-(\gamma+t)\sum_{i=1}^n x_i} dx_1 \ldots dx_n$$

$$= \int_0^\infty f_1(x_1) e^{-(\gamma+t)x_1} dx_1 \ldots \int_0^\infty f_n(x_n) e^{-(\gamma+t)x_n} dx_n. \tag{4.2}$$

Let us now bound each of these simple integrals. Let $p_{1,*}, \ldots p_{n,*} > 1$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_{1,*}} = 1, \quad \ldots, \quad \frac{1}{p_n} + \frac{1}{p_{n,*}} = 1,$$

so that

$$p_{1,*} = \frac{p_1}{p_1 - 1}, \quad \ldots, \quad p_{n,*} = \frac{p_n}{p_n - 1}.$$

For any $i = 1, \ldots, n$, applying the Hölder integral inequality with the parameters p_i and $p_{i,*}$, we get

$$\int_0^\infty f_i(x_i) e^{-(\gamma+t)x_i} dx_i \leq \|f_i\|_{p_i} \left(\int_0^\infty e^{-p_{i,*}(\gamma+t)x_i} dx_i \right)^{1/p_{i,*}} = \|f_i\|_{p_i} [p_{i,*}(\gamma + t)]^{-1/p_{i,*}}$$
$$= \|f_i\|_{p_i} p_{i,*}^{-1/p_{i,*}} (\gamma + t)^{-1/p_{i,*}}.$$

Multiplying these bounds and using

$$\sum_{i=1}^n \frac{1}{p_{i,*}} = \sum_{i=1}^n \left(1 - \frac{1}{p_i}\right) = n - \sum_{i=1}^n \frac{1}{p_i} = n - 1,$$

we obtain

$$\int_0^\infty f_1(x_1) e^{-(\gamma+t)x_1} dx_1 \ldots \int_0^\infty f_n(x_n) e^{-(\gamma+t)x_n} dx_n$$
$$\leq \|f_1\|_{p_1} \ldots \|f_n\|_{p_n} p_{1,*}^{-1/p_{1,*}} \ldots p_{n,*}^{-1/p_{n,*}} (\gamma + t)^{-\sum_{i=1}^n 1/p_{i,*}}$$
$$= \|f_1\|_{p_1} \ldots \|f_n\|_{p_n} p_{1,*}^{-1/p_{1,*}} \ldots p_{n,*}^{-1/p_{n,*}} (\gamma + t)^{-(n-1)}. \tag{4.3}$$

Combining Equations (4.1), (4.2) and (4.3), we get

$$\int_0^\infty \ldots \int_0^\infty \frac{e^{-\gamma \sum_{i=1}^n x_i}}{(\sum_{i=1}^n x_i)^\lambda} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n$$
$$\leq \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \|f_1\|_{p_1} \ldots \|f_n\|_{p_n} p_{1,*}^{-1/p_{1,*}} \ldots p_{n,*}^{-1/p_{n,*}} (\gamma + t)^{-(n-1)} dt$$
$$= \left(\frac{1}{\Gamma(\lambda)} p_{1,*}^{-1/p_{1,*}} \ldots p_{n,*}^{-1/p_{n,*}} \int_0^\infty t^{\lambda-1} (\gamma + t)^{-(n-1)} dt \right) \|f_1\|_{p_1} \ldots \|f_n\|_{p_n}. \tag{4.4}$$

Let us now focus on the simple integral with respect to t. Changing the variables $u = t/\gamma$ and using standard properties of the beta and gamma functions, we get

$$\int_0^\infty t^{\lambda-1} (\gamma + t)^{-(n-1)} dt = \gamma^{\lambda-(n-1)} \int_0^\infty u^{\lambda-1} (1 + u)^{-(n-1)} du$$
$$= \gamma^{\lambda-(n-1)} B(\lambda, n - 1 - \lambda) = \gamma^{\lambda-(n-1)} \frac{\Gamma(\lambda)\Gamma(n - 1 - \lambda)}{\Gamma(n - 1)}$$
$$= \gamma^{\lambda-(n-1)} \frac{1}{(n-2)!} \Gamma(\lambda)\Gamma(n - 1 - \lambda). \tag{4.5}$$

Combining Equations (4.4) and (4.5), simplifying $\Gamma(\lambda)$ and using the expressions of $p_{1,*}, \ldots, p_{n,*}$, we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{e^{-\gamma \sum_{i=1}^n x_i}}{(\sum_{i=1}^n x_i)^\lambda} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n$$
$$\leq \left(\frac{1}{\Gamma(\lambda)} p_{1,*}^{-1/p_{1,*}} \ldots p_{n,*}^{-1/p_{n,*}} \gamma^{\lambda-(n-1)} \frac{1}{(n-2)!} \Gamma(\lambda)\Gamma(n-1-\lambda) \right) \|f_1\|_{p_1} \ldots \|f_n\|_{p_n}$$
$$= \left(\left(\frac{p_1}{p_1-1} \right)^{1/p_1-1} \ldots \left(\frac{p_n}{p_n-1} \right)^{1/p_n-1} \gamma^{\lambda-(n-1)} \frac{1}{(n-2)!} \Gamma(n-1-\lambda) \right)$$
$$\times \|f_1\|_{p_1} \ldots \|f_n\|_{p_n}$$
$$= \Omega \|f_1\|_{p_1} \ldots \|f_n\|_{p_n},$$

which completes the proof. □

As an example, in dimension $n = 4$, for $\lambda \in (0, 3)$ and $\gamma > 0$, the following inequality holds:

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\gamma(x_1+x_2+x_3+x_4)}}{(x_1+x_2+x_3+x_4)^\lambda} f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4) dx_1 dx_2 dx_3 dx_4$$
$$\leq \Omega \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3} \|f_4\|_{p_4},$$

where

$$\Omega = \left(\frac{p_1}{p_1-1} \right)^{1/p_1-1} \left(\frac{p_2}{p_2-1} \right)^{1/p_2-1} \left(\frac{p_3}{p_3-1} \right)^{1/p_3-1} \left(\frac{p_4}{p_4-1} \right)^{1/p_4-1} \gamma^{\lambda-3} \frac{1}{2} \Gamma(3-\lambda).$$

Therefore, Theorem 4.1 generalizes Theorem 3.1 to n dimensions, establishing a Hardy-Hilbert-type integral inequality with an exponential-ratio kernel function. The constant Ω depends explicitly on the gamma function, the decay parameters λ and γ, and the exponents $p_1, \ldots, p_n$, reflecting the interplay between the singularity of the kernel function and its exponential decay.

As a particular case of interest, setting $\lambda = 1 \in (0, n-1)$, we get

$$\int_0^\infty \cdots \int_0^\infty \frac{e^{-\gamma \sum_{i=1}^n x_i}}{\sum_{i=1}^n x_i} f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n \leq \Omega \|f_1\|_{p_1} \ldots \|f_n\|_{p_n},$$

where

$$\Omega = \left(\frac{p_1}{p_1-1} \right)^{1/p_1-1} \ldots \left(\frac{p_n}{p_n-1} \right)^{1/p_n-1} \gamma^{2-n} \frac{1}{n-2}.$$

5. Conclusion

In this article, we presented a new three-dimensional Hardy-Hilbert-type integral inequality featuring an original exponential-ratio kernel function and a constant factor expressed in terms of the gamma function. We extended this approach to the multi-dimensional case, thereby demonstrating the versatility and applicability of the method. These results contribute to the ongoing development of Hardy-Hilbert-type integral inequalities and provide a framework for studying more general kernel functions in higher dimensions. Future research could investigate sharper bounds, alternative kernel functions and applications in functional analysis, fractional calculus and related inequalities in applied mathematics.

Conflict of interest

The author declares that there is no conflict of interest related to this paper.

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References

[1] L.E. Azar: *On some extensions of Hardy-Hilbert's inequality and applications*, J. Inequal. Appl., **2008** (2008), 1–14.

[2] L.E. Azar: *Two new forms of Hilbert integral inequality*, Math. Inequal. Appl., **17** (2014), 937–946.

[3] Q. Chen, B.C. Yang: *A survey on the study of Hilbert-type inequalities*, J. Inequal. Appl., **2015**:302 (2015).

[4] C. Chesneau: *Refining and extending two special Hardy-Hilbert-type integral inequalities*, Ann. Math. Comput. Sci., **28** (2025), 21–45.

[5] C. Chesneau: *Study of some new one-parameter modifications of the Hardy-Hilbert integral inequality*, Constr. Math. Anal., **8** (2025), 93–116.

[6] C. Chesneau: *Study of some new logarithmic-Hardy-Hilbert-type integral inequalities*, Earthline J. Math. Sci., **15** (2025), 837–848.

[7] C. Chesneau: *A new Hardy-Hilbert-type integral inequality involving an exponential-power kernel function*, *Adv. Math., Sci. J.*, **15** (2026), 77-87.

[8] G.H. Hardy, J.E. Littlewood, G. Pólya: *Inequalities*, Cambridge University Press, Cambridge, 1934.

[9] Y. LI, J. WU, B. HE: *A new Hilbert-type integral inequality and the equivalent form*, Int. J. Math. Math. Sci., **8** (2006), 1–6.

[10] A. SAGLAM, H. YILDIRIM, M.Z. SARIKAYA: *Generalization of Hardy-Hilbert's inequality and applications*, Kyungpook Math. J., **50** (2010), 131–152.

[11] M.Z. SARIKAYA, M.S. BINGOL: *Recent developments of integral inequalities of the Hardy-Hilbert type*, Turkish J. Inequal., **8** (2024), 43–54.

[12] W.T. SULAIMAN: *On three inequalities similar to Hardy-Hilbert's integral inequality*, Acta Math. Univ. Comenianae, **LXXVI** (2007), 273–278.

[13] W.T. SULAIMAN: *New Hardy-Hilbert's-type integral inequalities*, Int. Math. Forum, **3** (2008), 2139–2147.

[14] W.T. SULAIMAN: *Hardy-Hilbert's integral inequality in new kinds*, Math. Commun., **15** (2010), 453–461.

[15] W.T. SULAIMAN: *New kinds of Hardy-Hilbert's integral inequalities*, Appl. Math. Lett., **23** (2010), 361–365.

[16] W.T. SULAIMAN: *An extension of Hardy-Hilbert's integral inequality*, Afr. Diaspora J. Math., **10** (2010), 66–71.

[17] B. SUN: *Best generalization of a Hilbert type inequality*, J. Inequal. Pure Appl. Math., **7** (2006), 1–7.

[18] B.C. YANG: *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, United Arab Emirates, 2009.

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