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A NEW GENERALIZATION OF THE YOUNG INTEGRAL INEQUALITY

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ABSTRACT. This article presents a new generalization of the classical Young integral inequality, which is obtained by introducing an auxiliary function. The key innovation lies in the combination of the Chebyshev and Young integral inequalities. This creates a unified analytical framework that can be used to derive new bounds and relationships between monotonic functions and their inverses. This is demonstrated through several examples and related results.

1. INTRODUCTION

Inequalities play a fundamental role in mathematical analysis and its numerous applications. They provide powerful tools for estimating quantities and establishing relationships between functions. Comprehensive discussions and applications of various inequalities can be found in [2,3,14,16,29]. One of the most celebrated classical results in this field is the Young integral inequality, which was first introduced in [32]. This inequality elegantly characterizes the relationship between a function and its inverse through their integral representations. A formal statement is provided below.

Theorem 1.1 (Young integral inequality)**.** *Let* $c > 0$ *and* $f : [0, c] \to [0, +\infty)$ *be a continuous strictly increasing function satisfying* $f(0) = 0$*. Then, for any* $a \in [0, c]$

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and $b \in [0, f(c)]$, *we have*

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab,$$

where f^{-1} *denotes the inverse function of* f.

In recent years, there has been a growing interest in extending, refining and applying the Young integral inequality in various ways. Comprehensive surveys and related developments can be found in [1, 4–13, 15, 17–28, 30, 31, 33].

In this article, we address the topic of the Young integral inequality by presenting a new generalization involving an auxiliary function, g. This approach yields a broader class of inequalities that reduce to the classical Young integral inequality when $g(x) = 1$. The generalized form can be stated as follows:

$$a\left(\int_0^a g(x)dx\right)^{-1}\int_0^a f(x)g(x)dx + b\left(\int_0^b g(x)dx\right)^{-1}\int_0^b f^{-1}(x)g(x)dx \geq ab,$$

where the precise assumptions on the functions involved will be presented later. The key to this result lies in combining the Chebyshev and Young integral inequalities. This synthesis provides a flexible analytical framework that can be used to derive new bounds and relationships between monotonic functions and their inverses. Furthermore, it paves the way for further extensions and applications within the theory of integral inequalities and functional analysis. Several examples are discussed that illustrate specific choices of the function g, along with additional related results.

The remainder of this article is organized as follows: Section 2 presents the main result together with its proof and some examples. In Section 3, we discuss several additional results. Finally, Section 4 concludes the article with a summary of the main contributions and suggestions for future research.

2. Main result

2.1. **Statement and proof.**

The theorem below presents our generalization of the Young integral inequality. We emphasize the monotonic assumption on g, which is crucial to the proof.

Theorem 2.1. *Let* $c > 0$, $f : [0, c] \to [0, +\infty)$ *be a continuous strictly increasing function satisfying* $f(0) = 0$, *and* $g : [0, \max(c, f(c))] \to [0, +\infty)$ *be an increasing* *function. Then, for any* $a \in [0, c]$ *and* $b \in [0, f(c)]$*, we have*

$$a\left(\int_0^a g(x)dx\right)^{-1}\int_0^a f(x)g(x)dx + b\left(\int_0^b g(x)dx\right)^{-1}\int_0^b f^{-1}(x)g(x)dx \geq ab.$$

Proof. A key to the proof is the Chebyshev integral inequality, as recalled in the appendix of this article. Since both f and g are increasing functions, this inequality applied to f and g over the interval $[0, a]$ yields

$$\frac{1}{a-0}\int_0^a f(x)g(x)dx \geq \frac{1}{a-0}\int_0^a f(x)dx\frac{1}{a-0}\int_0^a g(x)dx,$$

which is equivalent to

$$\int_0^a f(x)g(x)dx \geq \frac{1}{a}\int_0^a f(x)dx\int_0^a g(x)dx. \tag{2.1}$$

Moreover, since f is increasing, its inverse function f^{-1} is also increasing. Hence, by applying the Chebyshev integral inequality to f^{-1} and g on the interval $[0, b]$, we obtain

$$\frac{1}{b-0}\int_0^b f^{-1}(x)g(x)dx \geq \frac{1}{b-0}\int_0^b f^{-1}(x)dx\frac{1}{b-0}\int_0^b g(x)dx,$$

which is equivalent to

$$\int_0^b f^{-1}(x)g(x)dx \geq \frac{1}{b}\int_0^b f^{-1}(x)dx\int_0^b g(x)dx. \tag{2.2}$$

Using Equations (2.1) and (2.2), the fact that g is non-negative implies that $\int_0^a g(x)dx$ and $\int_0^b g(x)dx$ are non-negative, and applying the classical Young integral inequality to f recalled in Theorem 1.1, we have

$$\begin{aligned}
&a\left(\int_0^a g(x)dx\right)^{-1}\int_0^a f(x)g(x)dx + b\left(\int_0^b g(x)dx\right)^{-1}\int_0^b f^{-1}(x)g(x)dx\\
&\geq a\left(\int_0^a g(x)dx\right)^{-1}\frac{1}{a}\int_0^a f(x)dx\int_0^a g(x)dx\\
&+ b\left(\int_0^b g(x)dx\right)^{-1}\frac{1}{b}\int_0^b f^{-1}(x)dx\int_0^b g(x)dx
\end{aligned}$$

$$= \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab.$$

This completes the proof. □

Clearly, if we set $g(x) = 1$, Theorem 2.1 reduces to the classical Young integral inequality for f, as stated in Theorem 1.1. The presence of g introduces a weighting mechanism that enables a more flexible and refined comparison between a function and its inverse. This enables a broader class of integral inequalities to be derived.

2.2. **Examples.** Some examples of Theorem 2.1 are now presented. The setting of this theorem is implicitly considered.

- If we take $g(x) = x^\alpha$, with $\alpha \geq 0$, Theorem 2.1 yields
$$a\left(\frac{a^{\alpha+1}}{\alpha+1}\right)^{-1}\int_0^a f(x)x^\alpha dx + b\left(\frac{b^{\alpha+1}}{\alpha+1}\right)^{-1}\int_0^b f^{-1}(x)x^\alpha dx \geq ab,$$
so that
$$a^{-\alpha}\int_0^a f(x)x^\alpha dx + b^{-\alpha}\int_0^b f^{-1}(x)x^\alpha dx \geq \frac{ab}{\alpha+1}.$$
If we set $\alpha = 0$, then the classical Young integral inequality is obtained. The other cases are new to the literature, as far as the author knows.
- If we take $g(x) = e^{\beta x}$, with $\beta \geq 0$, Theorem 2.1 yields
$$a\left(\frac{e^{\beta a}-1}{\beta}\right)^{-1}\int_0^a f(x)e^{\beta x}dx + b\left(\frac{e^{\beta b}-1}{\beta}\right)^{-1}\int_0^b f^{-1}(x)e^{\beta x}dx \geq ab,$$
so that
$$a\left(e^{\beta a}-1\right)^{-1}\int_0^a f(x)e^{\beta x}dx + b\left(e^{\beta b}-1\right)^{-1}\int_0^b f^{-1}(x)e^{\beta x}dx \geq \frac{ab}{\beta}.$$
If we set $\beta = 0$, then the classical Young integral inequality is obtained. The other cases are new to the literature, as far as the author knows.
- If we take $g(x) = \log(1+\gamma x)$, with $\gamma > 0$, Theorem 2.1 yields

$$a\left(\frac{1}{\gamma}(1+\gamma a)\log(1+\gamma a)-a\right)^{-1}\int_0^a f(x)\log(1+\gamma x)dx$$
$$+\,b\left(\frac{1}{\gamma}(1+\gamma b)\log(1+\gamma b)-b\right)^{-1}\int_0^b f^{-1}(x)\log(1+\gamma x)dx \geq ab.$$

Other examples can be found by using trigonometric or special functions for g.

3. Additional results

This section is devoted to additional results that, in one way or another, follow the spirit of Theorem 2.1.

3.1. **Consequences of Theorem 2.1.**

The proposition below presents a new one-function version of the Young integral inequality.

Proposition 3.1. *Let* $c > 0$ *and* $f : [0, c] \to [0, +\infty)$ *be a continuous strictly increasing function satisfying* $f(0) = 0$*. Then, for any* $a \in [0, c]$ *and* $b \in [0, \max(c, f(c))]$*, we have*

$$a\left(\int_0^a f(x)dx\right)^{-1}\int_0^a (f(x))^2 dx + b\left(\int_0^b f(x)dx\right)^{-1}\int_0^b f^{-1}(x)f(x)dx \geq ab.$$

Proof. Since the function f is increasing, the result follows directly from Theorem 2.1 by taking $g = f$. This completes the proof. □

The proposition below presents another new one-function version of the Young integral inequality.

Proposition 3.2. *Let* $c > 0$ *and* $f : [0, c] \to [0, +\infty)$ *be a continuous strictly increasing function satisfying* $f(0) = 0$*. Then, for any* $a \in [0, \max(c, f(c))]$ *and* $b \in [0, f(c)]$*, we have*

$$a\left(\int_0^a f^{-1}(x)dx\right)^{-1}\int_0^a f(x)f^{-1}(x)dx + b\left(\int_0^b f^{-1}(x)dx\right)^{-1}\int_0^b (f^{-1}(x))^2 dx \geq ab.$$

Proof. Since f is increasing, its inverse function f^{-1} is also increasing. The result follows directly from Theorem 2.1 by taking $g = f^{-1}$. This completes the proof. □

3.2. **Other general results.** Under the monotonic assumption on g, one can derive generalizations of the Young integral inequalities that are more direct than the one stated in Theorem 2.1, but with some limitations. Two such inequalities are presented below.

Theorem 3.1. *Let $c > 0$, $f : [0, c] \to [0, +\infty)$ be a continuous strictly increasing function satisfying $f(0) = 0$, and $g : [0, \max(c, f(c))] \to [0, +\infty)$ be a decreasing function. Then, for any $a \in [0, c]$ and $b \in [0, f(c)]$ such that $g(a) \neq 0$ and $g(b) \neq 0$, we have*

$$(g(a))^{-1} \int_0^a f(x)g(x)dx + (g(b))^{-1} \int_0^b f^{-1}(x)g(x)dx \geq ab.$$

Proof. Since f, f^{-1} and g are non-negative, and g is decreasing, we have

$$\int_0^a f(x)g(x)dx \geq g(a) \int_0^a f(x)dx \tag{3.1}$$

and

$$\int_0^b f^{-1}(x)g(x)dx \geq g(b) \int_0^b f^{-1}(x)dx. \tag{3.2}$$

Combining Equations (3.1) and (3.2), and applying the classical Young integral inequality to f recalled in Theorem 1.1, we have

$$\begin{aligned}
&(g(a))^{-1} \int_0^a f(x)g(x)dx + (g(b))^{-1} \int_0^b f^{-1}(x)g(x)dx \\
&\geq (g(a))^{-1} g(a) \int_0^a f(x)dx + (g(b))^{-1} g(b) \int_0^b f^{-1}(x)dx \\
&= \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab.
\end{aligned}$$

This completes the proof. □

The assumption that $g(a) \neq 0$ and $g(b) \neq 0$ limits the scope of this result. However, it has the advantage of being true for a decreasing function g, unlike Theorem 2.1.

The analogue of Theorem 3.1 is presented below, under the assumption that g is an increasing function.

Theorem 3.2. *Let $c > 0$, $f : [0, c] \to [0, +\infty)$ be a continuous strictly increasing function satisfying $f(0) = 0$, and $g : [0, \max(c, f(c))] \to [0, +\infty)$ be an increasing* *function. Then, for any* $a \in [0, c]$ *and* $b \in [0, f(c)]$*, we have*

$$\int_0^a f(x)g(x)dx + \int_0^b f^{-1}(x)g(x)dx \geq abg(0).$$

Proof. Since f, f^{-1} and g are non-negative, and g is increasing, we have

$$\int_0^a f(x)g(x)dx \geq g(0) \int_0^a f(x)dx \tag{3.3}$$

and

$$\int_0^b f^{-1}(x)g(x)dx \geq g(0) \int_0^b f^{-1}(x)dx. \tag{3.4}$$

Combining Equations (3.3) and (3.4), and applying the classical Young integral inequality to f recalled in Theorem 1.1, we have

$$\int_0^a f(x)g(x)dx + \int_0^b f^{-1}(x)g(x)dx \geq g(0) \int_0^a f(x)dx + g(0) \int_0^b f^{-1}(x)dx$$
$$= g(0) \left(\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \right) \geq abg(0).$$

This completes the proof. □

A notable limitation of this result occurs when $g(0) = 0$, which yields the trivial inequality

$$\int_0^a f(x)g(x)dx + \int_0^b f^{-1}(x)g(x)dx \geq 0,$$

since it is merely the sum of two non-negative integrals. This situation arises, for instance, in the key case $g(x) = x^\alpha$ with $\alpha > 0$. In such cases, Theorem 2.1 provides a more suitable formulation.

4. Conclusion

In conclusion, this study presents a new generalization of the classical Young integral inequality, incorporating an auxiliary function g. This methodology, which combines the Chebyshev and Young integral inequalities, provides a consistent analytical framework for deriving new bounds and relationships between monotonic functions and their inverses. Several illustrative examples and additional integral inequalities are presented. Promising avenues for future research include studying multidimensional extensions and operator formulations, and applying the methodology to problems in convex analysis and information theory.

Appendix

Due to its significance in the main proof, the theorem below provides a formal statement of the Chebyshev integral inequality.

Theorem 4.1 (Chebyshev integral inequality). *Let $a, b \in \mathbb{R}$ with $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be integrable functions that are both increasing or both decreasing. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

The inequality is reversed if one of the functions f or g is increasing and the other is decreasing.

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Conflict of interest

The author declares that there is no conflict of interest related to this paper.

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