Advances in Mathematics: Scientific Journal **15**(2) (2026), 183–190
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.15.2.1

RATE OF CONVERGENCE OF A FAMILY OF NON-CONVOLUTION INTEGRAL OPERATORS CONTAINING NON-INTEGRABLE FUNCTIONS AT LEBESGUE POINTS IN $L_1(a, b)$

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ABSTRACT. In this paper, we investigate the rate of convergence of a family of non-convolution type integral operators involving functions that do not belong to $L_1(a, b)$ at their Lebesgue points. Employing an auxiliary function $\varphi \in L_1(a, b)$ and suitable monotonicity and integrability conditions on the kernel $K_\lambda(t, x)$, we establish the quantitative estimate

1. INTRODUCTION

The study of pointwise convergence properties of families of integral operators is a classical and active topic in approximation theory. A key notion in this context is that of a *Lebesgue point*: a point x is a Lebesgue point of a locally integrable function g if

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h |g(x+t)-g(x)|\,dt \;=\; 0 \quad \text{and} \quad \lim_{h\to 0^+} \frac{1}{h}\int_0^h |g(x-t)-g(x)|\,dt \;=\; 0.$$

By the Lebesgue differentiation theorem, almost every point of (a, b) for a function in $L_1(a, b)$ is a Lebesgue point. Convergence of singular integrals and positive linear operators at such points was studied systematically by Butzer and Nessel

2020 *Mathematics Subject Classification.* Primary: 41A35, 41A25; Secondary: 47G10, 26A15.
Key words and phrases. Non-integrable function; non-convolution type integral operator family; rate of convergence at Lebesgue points; L_1 approximation.
Submitted: 27.03.2026; *Accepted:* 13.04.2026; *Published:* 23.04.2026.

[3], whose monograph provides the foundational framework for approximation in Lebesgue spaces. Precise asymptotic estimates for convolution-type operators satisfying one-sided monotonicity conditions were obtained by Taberski [6].

Many operators of practical interest, however, have kernels that depend separately on t and x rather than on the difference $t - x$, and are therefore not of convolution type. Such *non-convolution* operators appear prominently in the theory of positive linear operators; see Gadjiev [4] and Karsli [5] for foundational contributions to this setting. A further complication arises when the function being approximated does not belong to the underlying L_p space. The use of a multiplicative auxiliary function φ to handle such non-integrable inputs was employed in the L_p framework by Bardaro, Butzer, Stens and Vinti [2], and in the non-convolution setting by the author in [1].

The present paper focuses on the L_1 case. Let $f \notin L_1(a,b)$ and let $\varphi \in L_1(a,b)$ be an auxiliary function such that $f_\varphi := f/\varphi \in L_1(a,b)$. We consider the family of integral operators

$$L_\lambda(f;\, x) \;=\; \int_a^b f(t)\, K_\lambda(t,x)\, dt, \qquad \lambda > 0, \tag{1.1}$$

where $\{K_\lambda\}_{\lambda>0}$ is a non-negative kernel family satisfying conditions 1–5 below. The main result, Theorem 3.1, establishes the quantitative rate

$$\big|L_\lambda(f;\, x) - \big(\Theta(x)A(x) + \Psi(x)B(x)\big)\big| = o(\lambda^{-1}) \quad \text{as } \lambda \to \infty,$$

where $\Theta(x)$ and $\Psi(x)$ are one-sided limit values of f_φ at x, and $A(x)$, $B(x)$ are the one-sided weighted integrals of φ against the kernel. This result extends the qualitative approximation results of [1] to a quantitative rate estimate in $L_1(a,b)$.

The paper is organised as follows. Section 2 introduces the Lebesgue point condition and the kernel assumptions. Section 3 contains the full proof of the main theorem. Section 4 closes with remarks on extensions and open problems.

2. Preliminaries

We work on a compact interval $[a,b] \subset \mathbb{R}$. The space $L_1(a,b)$ is equipped with the norm $\|g\|_1 = \int_a^b |g(t)|\, dt$.

Definition 2.1 (Lebesgue point for non-integrable functions). *Let $\varphi \in L_1(a,b)$, $f \notin L_1(a,b)$, and suppose $f_\varphi := f/\varphi \in L_1(a,b)$. A point $x \in (a,b)$ is called a*

Lebesgue point *of f_φ with one-sided limit values $\Theta(x)$ (right) and $\Psi(x)$ (left) if there exists $\beta \geq 0$ such that*

$$\int_0^h \left| \frac{f(x+t)}{\varphi(x+t)} - \Theta(x) \right| dt = o(h^{\beta+1}), \tag{2.1}$$

$$\int_0^h \left| \frac{f(x-t)}{\varphi(x-t)} - \Psi(x) \right| dt = o(h^{\beta+1}), \tag{2.2}$$

as $h \to 0^+$.

The following conditions are imposed on the kernel family $\{K_\lambda\}_{\lambda>0}$ for a fixed $x \in (a,b)$.

(1) As a function of t, $K_\lambda(t,x)$ is non-decreasing on $[a,x]$ and non-increasing on $[x,b]$.

(2) For every $\delta > 0$ and $\beta \geq 0$,

$$\lambda_\delta \;:=\; \int_x^{x+\delta} |t-x|^\beta \, K_\lambda(t,x)\, dt \;\to\; 0 \quad \text{as } \lambda \to \infty$$

(and the symmetric condition holds on $[x-\delta, x]$).

(3) For every $\delta > 0$, $K_\lambda(x \pm \delta,\, x) = o(\lambda)$ as $\lambda \to \infty$.

(4) $A(x) := \int_a^x \varphi(t)\, K_\lambda(t,x)\, dt < \infty$ and $B(x) := \int_x^b \varphi(t)\, K_\lambda(t,x)\, dt < \infty$.

(5) The functions φ and $K_\lambda(\cdot, x)$ are almost everywhere differentiable in t, and $\varphi'(t)\, \partial_t K_\lambda(t,x) > 0$ a.e.

3. Main Result and Proof

Theorem 3.1. *Let $\beta \geq 0$, $\varphi \in L_1(a,b)$, $f \notin L_1(a,b)$, and $f_\varphi = f/\varphi \in L_1(a,b)$. Suppose the non-negative kernel family $\{K_\lambda\}$ satisfies conditions 1–5, and let $A(x)$ and $B(x)$ be as in 4. If x is a Lebesgue point of f_φ in the sense of Definition 2.1, then*

$$\bigl|L_\lambda(f;\, x) - \bigl(\Theta(x)A(x) + \Psi(x)B(x)\bigr)\bigr| \;=\; o(\lambda^{-1}) \quad \textit{as } \lambda \to \infty. \tag{3.1}$$

Proof. The proof proceeds in six steps.

Step 1: Choice of ε and δ. By the Lebesgue point conditions (2.1)–(2.2), for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \leq \delta$,

$$\int_0^h \left| \frac{f(x+t)}{\varphi(x+t)} - \Theta(x) \right| dt < \varepsilon\, h^{\beta+1}, \tag{3.2}$$

$$\int_0^h \left| \frac{f(x-t)}{\varphi(x-t)} - \Psi(x) \right| dt < \varepsilon\, h^{\beta+1}. \tag{3.3}$$

Step 2: Four-term decomposition. For the chosen δ and using condition 4, we write

$$\begin{aligned}
&\big|L_\lambda(f;\, x) - \big(\Theta(x)A(x) + \Psi(x)B(x)\big)\big| \\
&\le \int_a^{x-\delta} |f_\varphi(t) - \Psi(x)|\; \varphi(t)K_\lambda(t,x)\, dt \qquad (3.4)\\
&\quad + \int_{x-\delta}^{x} |f_\varphi(t) - \Psi(x)|\; \varphi(t)K_\lambda(t,x)\, dt \\
&\quad + \int_{x}^{x+\delta} |f_\varphi(t) - \Theta(x)|\; \varphi(t)K_\lambda(t,x)\, dt \qquad (3.5)\\
&\quad + \int_{x+\delta}^{b} |f_\varphi(t) - \Theta(x)|\; \varphi(t)K_\lambda(t,x)\, dt \\
&=: B_{1,\lambda} + B_{2,\lambda} + B_{3,\lambda} + B_{4,\lambda}. \qquad (3.6)
\end{aligned}$$

Step 3: Estimation of $B_{1,\lambda}$ and $B_{4,\lambda}$ (far-field terms). By condition 1, $K_\lambda(t,x)$ is non-decreasing on $[a,x]$, so $K_\lambda(t,x) \le K_\lambda(x-\delta,x)$ for all $t \in [a, x-\delta]$. By condition 5 and [1], φ is also non-decreasing on $[a, x-\delta]$, so

$$\begin{aligned}
B_{1,\lambda} &\le \int_a^{x-\delta} \Big(|f_\varphi(t)| + |\Psi(x)|\Big)\varphi(t)K_\lambda(t,x)\, dt \\
&\le \varphi(x-\delta)\, K_\lambda(x-\delta,x)\Big(\|f_\varphi\|_1 + |\Psi(x)|\,(b-a)\Big). \qquad (3.7)
\end{aligned}$$

Analogously, the non-increasing property of $K_\lambda(\cdot,x)$ on $[x,b]$ gives $K_\lambda(t,x) \le K_\lambda(x+\delta,x)$ for $t \in [x+\delta, b]$, and hence

$$B_{4,\lambda} \;\le\; \varphi(x+\delta)\, K_\lambda(x+\delta,x)\Big(\|f_\varphi\|_1 + |\Theta(x)|\,(b-a)\Big). \tag{3.8}$$

Step 4: Estimation of $B_{2,\lambda}$ (near-field, left side). Define the auxiliary cumulative function

$$G(t) \;:=\; \int_0^t \left| \frac{f(x-w)}{\varphi(x-w)} - \Psi(x) \right| dw, \qquad t \in [0,\delta]. \tag{3.9}$$

From (3.3), $G(t) < \varepsilon\, t^{\beta+1}$ for $t \in [0,\delta]$. Performing the substitution $u = x - t$ in $B_{2,\lambda}$ and then renaming $u \mapsto t$ gives

$$B_{2,\lambda} \;=\; \int_0^\delta \varphi(x-t)\, K_\lambda(x-t,x)\, dG(t). \tag{3.10}$$

Integration by parts yields

$$\begin{aligned} B_{2,\lambda} &= \Big[G(t)\,\varphi(x-t)K_\lambda(x-t,x)\Big]_0^\delta - \int_0^\delta G(t)\,\frac{d}{dt}\big[\varphi(x-t)K_\lambda(x-t,x)\big]\,dt \\ &= G(\delta)\,\varphi(x-\delta)K_\lambda(x-\delta,x) - \int_0^\delta G(t)\,\frac{d}{dt}\big[\varphi(x-t)K_\lambda(x-t,x)\big]\,dt, \end{aligned} \tag{3.11}$$

using $G(0) = 0$. By condition 5 the product $\varphi(x-t)K_\lambda(x-t,x)$ is non-increasing in t, so the derivative in the integrand is non-positive a.e. and therefore $-\frac{d}{dt}[\varphi(x-t)K_\lambda(x-t,x)] \geq 0$. Using $G(t) < \varepsilon\, t^{\beta+1}$,

$$B_{2,\lambda} \;\leq\; \varepsilon\,\delta^{\beta+1}\,\varphi(x-\delta)K_\lambda(x-\delta,x) + \varepsilon \int_0^\delta t^{\beta+1} \left| \frac{d}{dt}\big[\varphi(x-t)K_\lambda(x-t,x)\big] \right| dt. \tag{3.12}$$

Applying integration by parts a second time to the remaining integral,

$$\begin{aligned} \int_0^\delta t^{\beta+1} \left| \frac{d}{dt}[\varphi(x-t)K_\lambda(x-t,x)] \right| dt &= -\int_0^\delta t^{\beta+1}\,\frac{d}{dt}\big[\varphi(x-t)K_\lambda(x-t,x)\big]\,dt \\ &\leq (\beta+1)\int_0^\delta t^\beta\,\varphi(x-t)K_\lambda(x-t,x)\,dt. \end{aligned} \tag{3.13}$$

Reverting to the original variable via $u = x - t$ and then relabelling $u = t$,

$$B_{2,\lambda} \;\leq\; \varepsilon\,\delta^{\beta+1}\,\varphi(x-\delta)K_\lambda(x-\delta,x) + (\beta+1)\varepsilon\,\varphi(x)\int_{x-\delta}^x (x-t)^\beta K_\lambda(t,x)\,dt. \tag{3.14}$$

Step 5: Estimation of $B_{3,\lambda}$ (near-field, right side). Define

$$H(t) \;:=\; \int_0^t \left| \frac{f(x+u)}{\varphi(x+u)} - \Theta(x) \right| du, \qquad t \in [0,\delta]. \tag{3.15}$$

From (3.2), $H(t) < \varepsilon\, t^{\beta+1}$ for $t \in [0,\delta]$. With the substitution $u = t - x$ we write

$$B_{3,\lambda} = \int_0^\delta \varphi(x+t)\, K_\lambda(x+t,x)\, dH(t). \tag{3.16}$$

An identical integration-by-parts argument, using the non-increasing property of $\varphi(x+t)K_\lambda(x+t,x)$ in t (ensured by condition 5), gives

$$B_{3,\lambda} \;\le\; \varepsilon\,\delta^{\beta+1}\,\varphi(x+\delta)K_\lambda(x+\delta,x) + (\beta+1)\varepsilon\,\varphi(x)\int_x^{x+\delta}(t-x)^\beta K_\lambda(t,x)\,dt. \tag{3.17}$$

Adding (3.14) and (3.17),

$$\begin{aligned} B_{2,\lambda}+B_{3,\lambda} &\le \varepsilon\,\delta^{\beta+1}\big[\varphi(x-\delta)K_\lambda(x-\delta,x)+\varphi(x+\delta)K_\lambda(x+\delta,x)\big] \\ &\quad + (\beta+1)\varepsilon\,\varphi(x)\int_{x-\delta}^{x+\delta}|t-x|^\beta\,K_\lambda(t,x)\,dt. \end{aligned} \tag{3.18}$$

Step 6: Assembly and passage to the limit. Collecting (3.7), (3.8) and (3.18) into (3.6), and denoting by C_1, C_2, C_3 finite positive constants that depend on $\|f_\varphi\|_1$, $|\Theta(x)|$, $|\Psi(x)|$, $\varphi(x)$, β and $b-a$ but are independent of λ and ε, we obtain

$$|L_\lambda(f;\,x)-(\Theta(x)A(x)+\Psi(x)B(x))| \tag{3.19}$$

$$\le C_1\,\varphi(x-\delta)K_\lambda(x-\delta,x)+C_2\,\varphi(x+\delta)K_\lambda(x+\delta,x)$$

$$+\,C_3\,\varepsilon\int_{x-\delta}^{x+\delta}|t-x|^\beta\,K_\lambda(t,x)\,dt. \tag{3.20}$$

Dividing both sides by λ:

$$\frac{|L_\lambda(f;\,x)-(\Theta(x)A(x)+\Psi(x)B(x))|}{\lambda} \tag{3.21}$$

$$\le C_1\,\frac{\varphi(x-\delta)K_\lambda(x-\delta,x)}{\lambda}+C_2\,\frac{\varphi(x+\delta)K_\lambda(x+\delta,x)}{\lambda}$$

$$+\,C_3\,\varepsilon\,\frac{\displaystyle\int_{x-\delta}^{x+\delta}|t-x|^\beta\,K_\lambda(t,x)\,dt}{\lambda}. \tag{3.22}$$

By condition 3, $K_\lambda(x\pm\delta,x)/\lambda\to 0$ as $\lambda\to\infty$ for every fixed $\delta>0$. By condition 2, $\int_{x-\delta}^{x+\delta}|t-x|^\beta\,K_\lambda(t,x)\,dt\to 0$, so the third term on the right of (3.22) tends to zero as well. Taking $\lambda\to\infty$,

$$0 \;\le\; \limsup_{\lambda\to\infty}\frac{|L_\lambda(f;\,x)-(\Theta(x)A(x)+\Psi(x)B(x))|}{\lambda} \;\le\; 0.$$

Since $\varepsilon>0$ is arbitrary and $\varphi(x)$ is a finite constant,

$$\lim_{\lambda\to\infty}\frac{\big|L_\lambda(f;\,x)-\big(\Theta(x)A(x)+\Psi(x)B(x)\big)\big|}{\lambda} \;=\; 0,$$

which is precisely the statement

$$\left|L_\lambda(f;\,x) - \big(\Theta(x)A(x) + \Psi(x)B(x)\big)\right| = o(\lambda^{-1}),$$

completing the proof. ☐

4. Concluding Remarks

Theorem 3.1 establishes the rate $o(\lambda^{-1})$ for the L_1 setting, which is the natural counterpart of the $o(\lambda^{-1/p})$ rate for the L_p case ($p > 1$). Together, these results provide a complete picture of the convergence rates at generalised Lebesgue points for non-convolution operators acting on non-integrable functions.

Remark 4.1. *The constant $(\beta+1)\varphi(x)$ appearing in the near-field estimates (Steps 4 and 5) plays the role of the local oscillation of f_φ near x, and its finiteness is guaranteed by the Lebesgue point condition of Definition 2.1. Replacing the pointwise value $\varphi(x)$ by an L^1-averaged version would allow extension to functions φ that are merely locally integrable at x.*

Remark 4.2. *The proof uses only condition 5 to ensure the monotonicity of the products $\varphi(\cdot \pm t)K_\lambda(\cdot \pm t, x)$ in t, which is the key ingredient for the integration-by-parts steps. It would be of interest to weaken 5 to a one-sided or averaged positivity condition, thereby enlarging the class of admissible kernels.*

Remark 4.3. *An extension of the present results to unbounded intervals $[0,\infty)$ or $\mathbb{R}$ requires additional decay conditions on the kernel and will be addressed in forthcoming work.*

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