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TWO NEW AND SIMPLE IDEAS OF ONE-PARAMETER TRIGONOMETRIC UNIT DISTRIBUTIONS

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ABSTRACT. This article presents two new and simple ideas of unit distributions. These distributions are trigonometric in nature and depend on a single adjustable parameter. The main associated functions are derived in closed form. Several additional properties are also established, including moments and distributional results. General families of distributions are also introduced. Graphical illustrations support the theoretical findings.

1. INTRODUCTION

The probability and statistics literature proposes a wide variety of distributions of diverse natures. Among these, distributions supported on the interval $(0, 1)$, commonly referred to as unit distributions, play a fundamental role. Such distributions are particularly useful for modeling variables that represent probabilities, proportions, or rates. We refer the reader to the contemporary survey in [1], which presents in detail more than one hundred unit distributions, along with numerous relevant references.

In this article, we make a contribution to this field by introducing two new and simple ideas of unit distributions, in line with the approach taken in [3] and [5].

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They are of trigonometric nature and depend on a single adjustable parameter. For each of them, we derive the cumulative distribution function (CDF), the survival function (SF), the probability density function (PDF), and the quantile function (QF). We also investigate several moment properties and discuss how these distributions can be extended to generate more flexible and general families. In addition, graphical analyses are provided to illustrate the behavior of the CDFs and PDFs, thereby supporting their definitions and properties. These developments lay the basis for potential statistical applications, which we leave for future investigation by specialists in the field.

The remainder of the article is organized as follows: The first idea is presented in Section 2. Section 3 is devoted to the second idea. Finally, concluding remarks are given in Section 4.

2. First idea

2.1. **Main functions.** Our first idea is to create a unit distribution defined by a CDF that mixes the functionalities of the exponential and tangent functions in a simple manner, also with the use of an adjustable parameter. This CDF is described in the theorem below.

Theorem 2.1. *A valid CDF of a unit distribution is given by*

$$F(x) = \frac{1}{1 + e^{\lambda/\tan(\pi x)}}, \quad x \in (0,1),$$

with $\lambda > 0$*, which we complete with* $F(x) = 0$ *for any* $x \leq 0$ *and* $F(x) = 1$ *for any* $x \geq 1$.

Proof. Clearly, F is continuous on $\mathbb{R}\backslash\{0,1\}$. The points 0 and 1 need special treatment. For the point 0, we have

$$\lim_{x \to 0^+} \frac{\lambda}{\tan(\pi x)} = +\infty,$$

so that

$$\lim_{x \to 0^+} e^{\lambda/\tan(\pi x)} = +\infty$$

and

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{1}{1 + e^{\lambda/\tan(\pi x)}} = 0 = F(0).$$

F is thus continuous at 0.

For the point 1, we have

$$\lim_{x\to 1^-} \frac{\lambda}{\tan(\pi x)} = -\infty,$$

so that

$$\lim_{x\to 1^-} e^{\lambda/\tan(\pi x)} = 0$$

and

$$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^+} \frac{1}{1+e^{\lambda/\tan(\pi x)}} = 1 = F(1).$$

F is thus continuous at 1, ending the proof of the full continuity of F on $\mathbb{R}$.

Using standard derivation techniques, for any $x \in (0,1)$, we have

$$F'(x) = \lambda\pi \frac{e^{\lambda/\tan(\pi x)}}{(\sin(\pi x))^2 \left(1+e^{\lambda/\tan(\pi x)}\right)^2}.$$

It is clear that $F'(x) \geq 0$, demonstrating that F is increasing. We also obviously have $F(x) \in [0,1]$ for any $x \in \mathbb{R}$. As a result, F is a valid CDF of a unit distribution. This ends the proof. □

Let us call the unit distribution defined by F in Theorem 2.1 the exponential-tangent (E-T) distribution. It thus depends on a single parameter $\lambda > 0$.

For illustrative purposes, Figure 1 presents the curves of F for several values of λ.

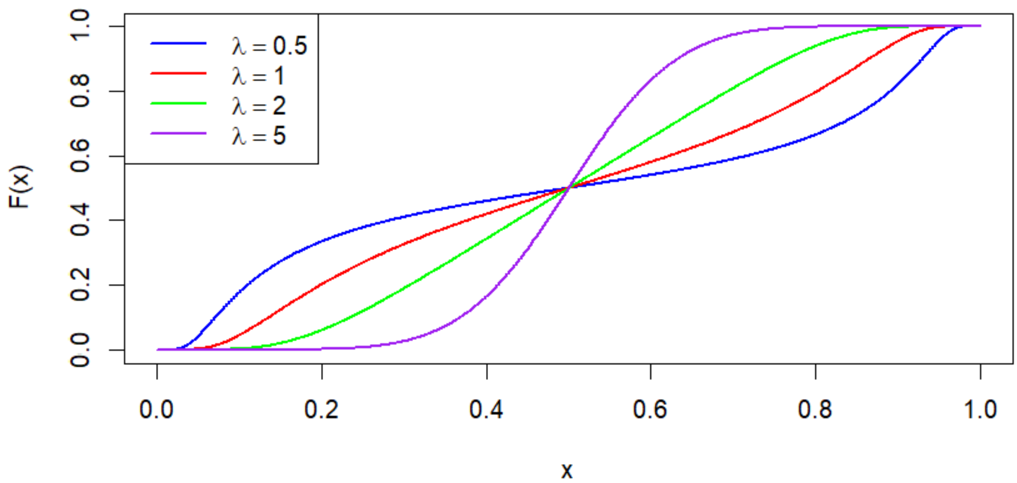

FIGURE 1. Curves of the CDF of the E-T distribution for several values of λ.

Various concave and convex shapes are observed, revealing the flexibility of the CDF.

The SF associated with the E-T distribution is expressed in the theorem below.

Theorem 2.2. *The SF associated with the E-T distribution is given by*

$$S(x) = \frac{1}{1 + e^{-\lambda/\tan(\pi x)}}, \quad x \in (0, 1),$$

with $\lambda > 0$*, which we complete with* $S(x) = 1$ *for any* $x \leq 0$ *and* $S(x) = 0$ *for any* $x \geq 1$.

Proof. Since the E-T distribution is a unit distribution, we have $S(x) = 1$ for any $x \leq 0$ and $S(x) = 0$ for any $x \geq 1$. For any $x \in (0, 1)$, using the expression of the CDF F in Theorem 2.1, we have

$$\begin{aligned} S(x) = 1 - F(x) &= 1 - \frac{1}{1 + e^{\lambda/\tan(\pi x)}} \\ &= \frac{e^{\lambda/\tan(\pi x)}}{1 + e^{\lambda/\tan(\pi x)}} = \frac{1}{1 + e^{-\lambda/\tan(\pi x)}}. \end{aligned}$$

This concludes the proof. □

The PDF associated with the E-T distribution is expressed in the theorem below.

Theorem 2.3. *The PDF associated with the E-T distribution is given by*

$$f(x) = \lambda\pi \frac{e^{\lambda/\tan(\pi x)}}{(\sin(\pi x))^2 \left(1 + e^{\lambda/\tan(\pi x)}\right)^2}, \quad x \in (0, 1),$$

with $\lambda > 0$*, which we complete with* $f(x) = 0$ *for any* $x \notin (0, 1)$.

Proof. Since the E-T distribution is a unit distribution, we have $f(x) = 0$ for any $x \notin (0, 1)$. For any $x \in (0, 1)$, using the expression of the CDF F in Theorem 2.1 and standard derivation techniques, we have

$$f(x) = F'(x) = \lambda\pi \frac{e^{\lambda/\tan(\pi x)}}{(\sin(\pi x))^2 \left(1 + e^{\lambda/\tan(\pi x)}\right)^2}.$$

This concludes the proof. □

For illustrative purposes, Figure 2 presents the curves of f for several values of λ.

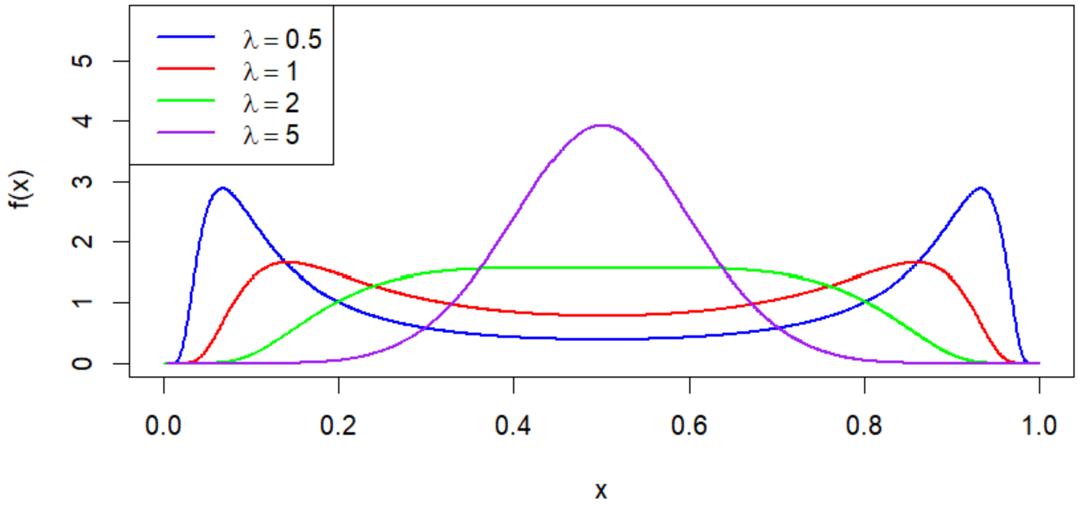

FIGURE 2. Curves of the PDF of the E-T distribution for several values of λ.

Bimodal and unimodal curves are observed, with a symmetry with respect to $x = 1/2$. This flexibility validates the interest of the E-T distribution.

The QF associated with the E-T distribution is expressed in the theorem below.

Theorem 2.4. *The QF associated with the E-T distribution is given by*

$$Q(u) = \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right), \quad u \in \left(0, \frac{1}{2}\right),$$

$Q(1/2) = 1/2$ *and*

$$Q(u) = \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right) + 1, \quad u \in \left(\frac{1}{2}, 1\right),$$

with $\lambda > 0$.

Proof. The QF associated with the E-T distribution corresponds to the inverse of the CDF F, as described in Theorem 2.1. Therefore, we need to solve the equation $F(x) = u$, which yields the following equivalences:

$$F(x) = u \Leftrightarrow \frac{1}{1 + e^{\lambda/\tan(\pi x)}} = u \Leftrightarrow e^{\lambda/\tan(\pi x)} = \frac{1}{u} - 1$$
$$\Leftrightarrow \frac{\lambda}{\tan(\pi x)} = \log\left(\frac{1}{u} - 1\right) \Leftrightarrow \tan(\pi x) = \frac{\lambda}{\log(1/u - 1)}.$$

For $u \in (0, 1/2)$, we have

$$x = \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right).$$

For $u \to 1/2$, we have a limit case giving $Q(1/2) = 1/2$. For $u \in (1/2, 1)$, taking into account the periodicity of the tangent function, we have

$$\begin{aligned} x &= \frac{1}{\pi}\left(\arctan\left(\frac{\lambda}{\log(1/u - 1)}\right) + \pi\right) \\ &= \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right) + 1. \end{aligned}$$

Therefore, we have

$$Q(u) = \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right), \quad u \in \left(0, \frac{1}{2}\right),$$

$Q(1/2) = 1/2$ and

$$Q(u) = \frac{1}{\pi} \arctan\left(\frac{\lambda}{\log(1/u - 1)}\right) + 1, \quad u \in \left(\frac{1}{2}, 1\right).$$

This completes the proof. □

In particular, the median associated with the E-T distribution is given by

$$\text{med} = Q\left(\frac{1}{2}\right) = \frac{1}{2}.$$

2.2. **Complemenary results.** A distribution result is proposed in the theorem below.

Theorem 2.5. *Let X be a random variable that follows the E-T distribution. Then $Y = 1 - X$ also follows the E-T distribution (with the same parameter).*

Proof. Let F_Y be the CDF of Y and F be the CDF of the E-T distribution as defined in Theorem 2.1. Since the E-T distribution is a unit distribution, the distribution of Y is also a unit distribution, which implies that $F_Y(x) = 0$ for any $x \leq 0$ and $F_Y(x) = 1$ for any $x \geq 1$, so that $F_Y(x) = F(x)$ for any $x \notin (0, 1)$. For any $x \in (0, 1)$, using the formula $\tan(\pi(1-x)) = -\tan(\pi x)$, we have

$$\begin{aligned} F_Y(x) &= P(Y \leq x) = P(1 - X \leq x) = P(X \geq 1 - x) = 1 - P(X < 1 - x) \\ &= 1 - F(1-x) = 1 - \frac{1}{1 + e^{\lambda/\tan(\pi(1-x))}} = 1 - \frac{1}{1 + e^{-\lambda/\tan(\pi x)}} \\ &= \frac{e^{-\lambda/\tan(\pi x)}}{1 + e^{-\lambda/\tan(\pi x)}} = \frac{1}{1 + e^{\lambda/\tan(\pi x)}} = F(x). \end{aligned}$$

Therefore, Y and X follow the same E-T distribution. This concludes the proof. □

The mean associated with the E-T distribution is determined in the theorem below.

Theorem 2.6. *Let X be a random variable that follows the E-T distribution. Then we have*

$$E(X) = \frac{1}{2}.$$

Proof. It follows from Theorem 2.5 that the random variables X and $1 - X$ follow the same E-T distribution. Based on this, we have

$$E(X) = E(1 - X) \Leftrightarrow E(X) = 1 - E(X) \Leftrightarrow 2E(X) = 1 \Leftrightarrow E(X) = \frac{1}{2}.$$

This completes the proof. □

No closed-form expressions are available for the other moments of the E-T distribution. However, they can be computed numerically without difficulty.

2.3. **Family of distributions.** As for any unit distribution, the E-T distribution can be used as a generator of distributions. See [4]. The theorem below formalizes the result.

Theorem 2.7. *Let G be the CDF of a continuous distribution. Based on G and the CDF F of the E-T distribution, the following function is a valid CDF:*

$$F_{\dagger}(x) = F(G(x)) = \frac{1}{1 + e^{\lambda/\tan(\pi G(x))}}, \quad x \in \mathbb{R},$$

with $\lambda > 0$.

Proof. Since $G(x) \in [0, 1]$ for any $x \in \mathbb{R}$ and F is the CDF of a unit distribution, the composition $F_{\dagger}(x) = F(G(x))$ is valid from the mathematical point of view. Moreover, the composition of two continuous functions is a continuous function, the composition of two increasing functions is an increasing function and clearly $F_{\dagger}(x) \in [0,1]$ for any $x \in \mathbb{R}$ because $F(x) \in [0,1]$. Therefore, $F_{\dagger}$ is a valid CDF. This completes the proof. $\square$

We call the family of distributions defined with the CDF $F_{\dagger}$ in Theorem 2.7 the E-T family of distributions.

Three notable members of this family with distinct supports are described below.

First member: We introduce the E-T power distribution as the distribution defined by the following CDF:

$$F_{\vee}(x) = \frac{1}{1 + e^{\lambda/\tan(\pi x^{\theta})}}, \quad x \in (0,1),$$

with $\lambda > 0$ and $\theta > 0$, and which we complete with $F_{\vee}(x) = 0$ for any $x \leq 0$ and $F_{\vee}(x) = 1$ for any $x \geq 1$. In this case, Theorem 2.7 was applied to $G(x) = x^{\theta}$ for any $x \in (0,1)$, and which we complete with $G(x) = 0$ for any $x \leq 0$ and $G(x) = 1$ for any $x \geq 1$; this is the CDF of the power distribution with the parameter θ.

Second member: We introduce the E-T Weibull distribution as the distribution defined by the following CDF:

$$F_{\triangle}(x) = \frac{1}{1 + e^{\lambda/\tan\left(\pi(1-e^{-(x/\beta)^{\alpha}})\right)}}, \quad x > 0,$$

with $\lambda > 0$, $\alpha > 0$ and $\beta > 0$, and which we complete with $F_{\triangle}(x) = 0$ for any $x \leq 0$. In this case, Theorem 2.7 was applied to $G(x) = 1 - e^{-(x/\beta)^{\alpha}}$ for any $x > 0$, and which we complete with $G(x) = 0$ for any $x \leq 0$; this is the CDF of the Weibull distribution with the parameters α and β.

Third member: We introduce the E-T logistic distribution as the distribution defined by the following CDF:

$$F_{\ddagger}(x) = \frac{1}{1 + e^{\lambda/\tan(\pi/(1+e^{-(x-\mu)/\sigma}))}}, \quad x \in \mathbb{R},$$

with $\lambda > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. In this case, Theorem 2.7 was applied to $G(x) = 1/(1 + e^{-(x-\mu)/\sigma})$ for any $x \in \mathbb{R}$, which is the CDF of the logistic distribution with the parameters μ and σ.

Each of these members can be studied independently, with potential applications in statistical analysis, including data analysis and regression modeling.

3. Second idea

3.1. **Main functions.** The second idea is inspired by the one-parameter arctangent unit distribution introduced by [2]. It is given by the following CDF:

$$F(x) = \frac{4}{\pi}\arctan(x), \quad x \in (0,1),$$

which we complete with $F(x) = 0$ for any $x \le 0$ and $F(x) = 1$ for any $x \ge 1$. This unit distribution is of interest because it exploits the functionalities of the arctangent distribution in a very simple manner. In [2], numerous applications support this claim, mainly with the creation of a more general family of distributions, the arctan-X family of distributions.

Our second idea is to propose a different unit distribution but with a different CDF, which still exploits the arctangent distribution and incorporates an adjustable parameter. This CDF is described in the theorem below.

Theorem 3.1. *A valid CDF of a unit distribution is given by*

$$F(x) = \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right), \quad x \in (0,1),$$

with $\alpha > 0$, which we complete with $F(x) = 0$ for any $x \le 0$ and $F(x) = 1$ for any $x \ge 1$.

Proof. Clearly, F is continuous on $\mathbb{R}\backslash\{0,1\}$. The points 0 and 1 need special treatment. For the point 0, we have

$$\lim_{x\to 0^+} \tan\left(\frac{\pi}{2}x\right) = 0,$$

so that

$$\lim_{x\to 0^+} \left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha} = 0$$

and

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right) = 0 = F(0).$$

F is thus continuous at 0.

For the point 1, we have

$$\lim_{x\to 1^-} \tan\left(\frac{\pi}{2}x\right) = +\infty,$$

so that

$$\lim_{x\to 1^-} \left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha} = +\infty$$

and

$$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^+} \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right) = \frac{2}{\pi} \times \frac{\pi}{2} = 1 = F(1).$$

F is thus continuous at 1, ending the proof of the full continuity of F on $\mathbb{R}$.

Using standard derivation techniques, for any $x \in (0,1)$, we have

$$F'(x) = \alpha \frac{(\tan((\pi/2)x))^{\alpha-1}}{(\cos((\pi/2)x))^2 \left(1 + (\tan((\pi/2)x))^{2\alpha}\right)}.$$

It is clear that $F'(x) \geq 0$, demonstrating that F is increasing. We also obviously have $F(x) \in [0,1]$ for any $x \in \mathbb{R}$. As a result, F is a valid CDF of a unit distribution. This concludes the proof. □

Let us call the unit distribution defined by F in Theorem 3.1 the arctangent-tangent (A-T) distribution. It thus depends on a single parameter $\alpha > 0$.

As a notable property, if we take $\alpha = 1$, then we have

$$F(x) = \frac{2}{\pi} \arctan\left(\tan\left(\frac{\pi}{2}x\right)\right) = x, \quad x \in (0,1),$$

which we complete with $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$. We recognize the CDF of the uniform distribution with support $(0,1)$. The other values of α gives a completely new CDF, to the best of our knowledge.

For illustrative purposes, Figure 3 presents the curves of the CDF F for several values of α.

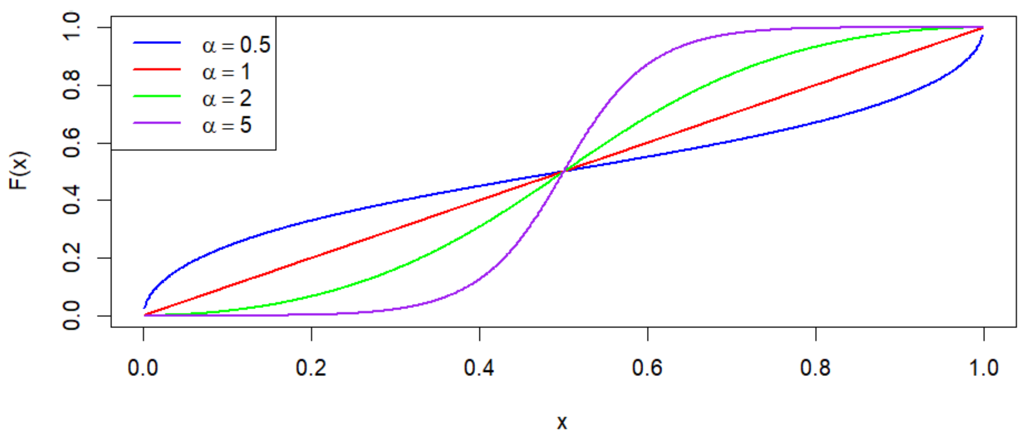

FIGURE 3. Curves of the CDF of the A-T distribution for several values of α.

Various concave and convex shapes are observed, revealing the flexibility of the CDF.

The SF associated with the A-T distribution is expressed in the theorem below.

Theorem 3.2. *The SF associated with the A-T distribution is given by*

$$S(x) = \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{-\alpha}\right), \quad x \in (0,1),$$

with $\alpha > 0$*, which we complete with* $S(x) = 1$ *for any* $x \leq 0$ *and* $S(x) = 0$ *for any* $x \geq 1$.

Proof. Since the A-T distribution is a unit distribution, we have $S(x) = 1$ for any $x \leq 0$ and $S(x) = 0$ for any $x \geq 1$. For any $x \in (0,1)$, using the expression of the CDF F in Theorem 3.1 and the formula $\arctan(y) + \arctan(1/y) = \pi/2$ for any $y > 0$, we have

$$\begin{aligned} S(x) = 1 - F(x) &= 1 - \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right) \\ &= \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right)\right) \\ &= \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{-\alpha}\right). \end{aligned}$$

This concludes the proof. □

The PDF associated with the A-T distribution is expressed in the theorem below.

Theorem 3.3. *The PDF associated with the A-T distribution is given by*

$$f(x) = \alpha\frac{(\tan((\pi/2)x))^{\alpha-1}}{(\cos((\pi/2)x))^2\left(1 + (\tan((\pi/2)x))^{2\alpha}\right)}, \quad x \in (0,1),$$

with $\alpha > 0$*, which we complete with* $f(x) = 0$ *for any* $x \notin (0,1)$.

Proof. Since the A-T distribution is a unit distribution, we have $f(x) = 0$ for any $x \notin (0,1)$. For any $x \in (0,1)$, using the expression of the CDF F in Theorem 3.1 and standard derivation techniques, we have

$$f(x) = F'(x) = \alpha\frac{(\tan((\pi/2)x))^{\alpha-1}}{(\cos((\pi/2)x))^2\left(1 + (\tan((\pi/2)x))^{2\alpha}\right)}.$$

This concludes the proof. □

For illustrative purposes, Figure 4 presents the curves of f for several values of α.

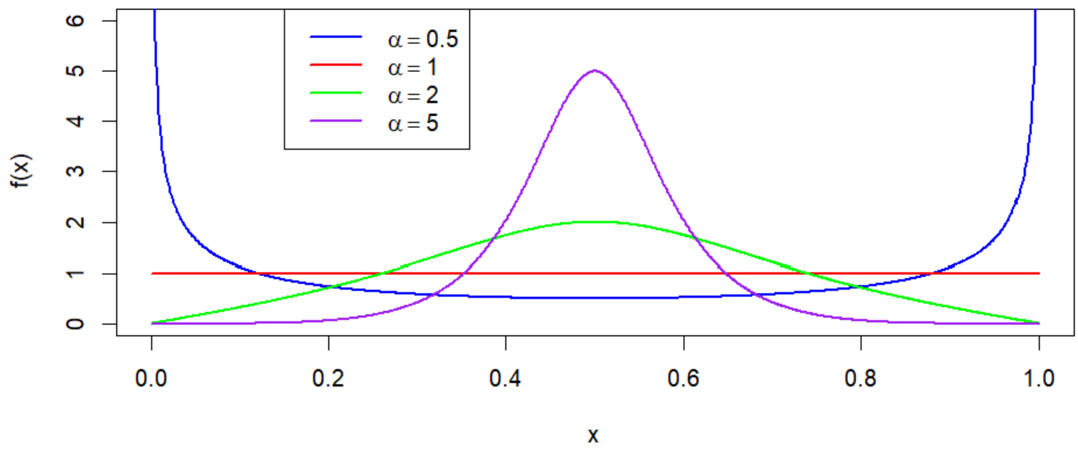

FIGURE 4. Curves of the PDF of the A-T distribution for several values of α.

Various unimodal and bathtub shapes are observed, with a symmetry with respect to $x = 1/2$. This flexibility validates the interest of the A-T distribution.

The QF associated with the A-T distribution is expressed in the theorem below.

Theorem 3.4. *The QF associated with the A-T distribution is given by*

$$Q(u) = \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}u\right)\right)^{1/\alpha}\right), \quad u \in (0,1),$$

with $\alpha > 0$.

Proof. The QF associated with the A-T distribution corresponds to the inverse of the CDF F, as described in Theorem 3.1. Therefore, we need to solve the equation $F(x) = u$, which yields the following equivalences:

$$F(x) = u \;\Leftrightarrow\; \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right) = u \;\Leftrightarrow\; \left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha} = \tan\left(\frac{\pi}{2}u\right)$$
$$\Leftrightarrow\; \tan\left(\frac{\pi}{2}x\right) = \left(\tan\left(\frac{\pi}{2}u\right)\right)^{1/\alpha} \;\Leftrightarrow\; x = \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}u\right)\right)^{1/\alpha}\right).$$

Therefore, we have

$$Q(u) = \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}u\right)\right)^{1/\alpha}\right), \quad u \in (0,1).$$

This completes the proof. □

In particular, the median associated with the A-T distribution is given by

$$\text{med} = Q\left(\frac{1}{2}\right) = \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{4}\right)\right)^{1/\alpha}\right) = \frac{2}{\pi}\times\frac{\pi}{4} = \frac{1}{2}.$$

3.2. **Complementary results.** A distribution result is proposed in the theorem below.

Theorem 3.5. *Let X be a random variable that follows the A-T distribution. Then $Y = 1 - X$ also follows the A-T distribution (with the same parameter).*

Proof. Let F_Y be the CDF of Y and F be the CDF of the A-T distribution as defined in Theorem 3.1. Since the A-T distribution is a unit distribution, the distribution of Y is also a unit distribution, which implies that $F_Y(x) = 0$ for any $x \leq 0$ and $F_Y(x) = 1$ for any $x \geq 1$, so that $F_Y(x) = F(x)$ for any $x \notin (0,1)$. For any $x \in (0,1)$, using the formulas $\tan((\pi/2)(1-x)) = 1/\tan((\pi/2)x)$ and $\arctan(y) + \arctan(1/y) = \pi/2$ for any $y > 0$, we have

$$\begin{aligned}
F_Y(x) &= P(Y \leq x) = P(1 - X \leq x) = P(X \geq 1 - x) = 1 - P(X < 1 - x) \\
&= 1 - F(1-x) = 1 - \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}(1-x)\right)\right)^{\alpha}\right) \\
&= 1 - \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{-\alpha}\right) \\
&= 1 - \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right)\right) \\
&= \frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x\right)\right)^{\alpha}\right) = F(x).
\end{aligned}$$

Therefore, Y and X follow the same A-T distribution. This concludes the proof. □

The mean associated with the A-T distribution is determined in the theorem below.

Theorem 3.6. *Let X be a random variable that follows the A-T distribution. Then we have*

$$E(X) = \frac{1}{2}.$$

Proof. It follows from Theorem 3.5 that the random variables X and $1-X$ follow the same A-T distribution. Based on this, we have

$$E(X)=E(1-X) \Leftrightarrow E(X)=1-E(X) \Leftrightarrow 2E(X)=1 \Leftrightarrow E(X)=\frac{1}{2}.$$

This completes the proof. □

No closed-form expressions are available for the other moments of the A-T distribution. However, they can be computed numerically without difficulty.

3.3. **Family of distributions.** As for any unit distribution, the A-T distribution can be used as a generator of distributions. See again [4]. The theorem below formalizes the result.

Theorem 3.7. *Let G be the CDF of a continuous distribution. Based on G and the CDF F of the A-T distribution, the following function is a valid CDF:*

$$F_{\dagger}(x)=F(G(x))=\frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}G(x)\right)\right)^{\alpha}\right), \quad x\in\mathbb{R},$$

with $\alpha>0$.

Proof. Since $G(x)\in[0,1]$ for any $x\in\mathbb{R}$ and F is the CDF of a unit distribution, the composition $F_{\dagger}(x)=F(G(x))$ is valid from the mathematical point of view. Moreover, the composition of two continuous functions is a continuous function, the composition of two increasing functions is an increasing function and clearly $F_{\dagger}(x)\in[0,1]$ for any $x\in\mathbb{R}$ because $F(x)\in[0,1]$. Therefore, $F_{\dagger}$ is a valid CDF. This completes the proof. □

We call the family of distributions defined with the CDF $F_{\dagger}$ in Theorem 3.7 the A-T family of distributions.

Three notable members of this family with distinct supports are described below.

First member: We introduce the A-T power distribution as the distribution defined by the following CDF:

$$F_{\vee}(x)=\frac{2}{\pi}\arctan\left(\left(\tan\left(\frac{\pi}{2}x^{\theta}\right)\right)^{\alpha}\right), \quad x\in(0,1),$$

with $\alpha>0$ and $\theta>0$, and which we complete with $F_{\vee}(x)=0$ for any $x\leq 0$ and $F_{\vee}(x)=1$ for any $x\geq 1$. In this case, Theorem 3.7 was applied to $G(x)=x^{\theta}$ for any $x\in(0,1)$, and which we complete with $G(x)=0$ for any $x\leq 0$ and $G(x)=1$ for any $x\geq 1$.

Second member: We introduce the A-T Weibull distribution as the distribution defined by the following CDF:

$$F_{\triangle}(x) = \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}\left(1 - e^{-(x/\beta)^{\tau}}\right)\right)\right)^{\alpha}\right), \quad x > 0,$$

with $\alpha > 0$, $\tau > 0$ and $\beta > 0$, and which we complete with $F_{\triangle}(x) = 0$ for any $x \leq 0$. In this case, Theorem 3.7 was applied to $G(x) = 1 - e^{-(x/\beta)^{\tau}}$ for any $x > 0$, and which we complete with $G(x) = 0$ for any $x \leq 0$.

Third member: We introduce the A-T logistic distribution as the distribution defined by the following CDF:

$$F_{\ddagger}(x) = \frac{2}{\pi} \arctan\left(\left(\tan\left(\frac{\pi}{2}\left(\frac{1}{1 + e^{-(x-\mu)/\sigma}}\right)\right)\right)^{\alpha}\right), \quad x \in \mathbb{R},$$

with $\alpha > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. In this case, Theorem 3.7 was applied to $G(x) = 1/(1 + e^{-(x-\mu)/\sigma})$ for any $x \in \mathbb{R}$.

Again, each of these members can be studied independently, with potential applications in statistical analysis, including data analysis and regression modeling.

4. Conclusion

This article introduces two new ideas of unit distributions with simple trigonometric structures and tractable closed-form expressions. Their main properties are established. The proposed distributions offer flexibility for statistical applications, particularly in data analysis and regression modeling. Future research may explore extensions to more general families, develop inference procedures, and investigate their performance in practical applications and real-world datasets.

Conflict of interest

The authors declare no competing interests.

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References

[1] E. Afuecheta, I.E. Okorie, H. Jallow, S. Nadarajah: *A review of unit continuous probability distributions*, AIMS Math., **10** (2025), 25939–26057.

[2] I. Alkhairy, M. Nagy, A.H. Muse, E. Hussam: *The Arctan-X family of distributions: properties, simulation, and applications to actuarial sciences*, Complexity, **2021**, Article ID 4689010, 14 pages.

[3] C. Chesneau, A. Artault: *On a comparative study on some trigonometric classes of distributions by the analysis of practical data sets*, J. Nonlinear Model. Anal., **3** (2021), 225–262.

[4] G.M. Cordeiro, R.B. Silva, A.D.C. Nascimento: *Recent Advances in Lifetime and Reliability Models*, Bentham Science Publishers, 2020.

[5] L. Souza, W.R.O. Júnior, C.C.R. de Brito, C. Chesneau, R.L. Fernandes, T.A.E. Ferreira: *Tan-G class of trigonometric distributions and its applications*, Cubo, **23** (2021), 1–20.

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