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EXTENSIONS OF SOME HÖLDER-TYPE INTEGRAL INEQUALITIES

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ABSTRACT. Hölder-type integral inequalities are inequalities that bound the integral of a product of functions by a certain operation of their norms. In this article, we present two extensions to theorems concerning these inequalities. These extensions are obtained using two intermediate weight functions. The proofs are provided in full for clarity and pedagogical completeness.

1. INTRODUCTION

The classical Hölder integral inequality is formally stated below. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $b > a$. Let $p > 1$ and $q = p/(p-1)$ be the Hölder conjugate. Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that $|f|^p$ and $|g|^q$ are integrable. Then we have

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}.$$

The Hölder integral inequality has been extensively generalized and refined in various directions. Significant contributions to this topic can be found in [1–13].

For the purposes of this article, we concentrate primarily on the seminal works [4, 5], which together have been cited over 250 times as of 2026. In particular, we recall [4, Theorem 2.1], which is stated below. Let $a, b \in \mathbb{R}$ with $b > a$. Let $p > 1$

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and $q = p/(p-1)$. Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that $|f|^p$ and $|g|^q$ are integrable.

- Then we have

$$\begin{aligned} &\int_a^b |f(x)g(x)|dx \\ &\le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{1/q}\right. \\ &\left.+\left(\int_a^b (x-a)|f(x)|^p dx\right)^{1/p}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{1/q}\right\}. \end{aligned}$$

- Then we have

$$\begin{aligned} &\frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{1/q}\right. \\ &\left.+\left(\int_a^b (x-a)|f(x)|^p dx\right)^{1/p}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{1/q}\right\} \\ &\le \left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q}. \end{aligned}$$

The first result has been extended in [4, Theorem 2.2] by introducing two weight functions whose sum is equal to 1. This was illustrated with trigonometric weight functions, using the well-known relation $\sin(x)^2 + \cos(x)^2 = 1$. Furthermore, a modification of [4, Theorem 2.1] was proposed in [5, Theorem 2.1], where it is presented as an improved power-mean integral inequality.

In this article, we establish two theorems that extend [4, Theorem 2.1] and [5, Theorem 2.1] by employing two intermediate weight functions whose sum is not necessarily equal to 1. We thus deal with the weighted integral norms of the forms

$$\int_a^b w(x)|f(x)g(x)|dx, \quad \int_a^b w(x)|f(x)|dx, \quad \int_a^b w(x)|g(x)|dx,$$

where $w : [a,b] \to [0,+\infty)$ denotes a certain weight function.

The proofs follow arguments similar to those used in [4, Theorem 2.1] and [5, Theorem 2.1], with appropriate modifications to handle the weight functions. These results are of particular interest due to their generality, as they can be adapted to a wide range of situations, including those considered in [4, 5]. For completeness and clarity, the proofs are presented in full detail.

The remainder of this article is organized as follows: Sections 2 and 3 are devoted to the presentation of our two main theorems. Concluding remarks are provided in Section 4.

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2. First theorem

Building on the work of [4, Theorem 2.1], the theorem below presents a generalized Hölder-type integral inequality involving two weight functions.

Theorem 2.1. *Let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *with* $b > a$. *Let* $p > 1$ *and* $q = p/(p-1)$. *Let* $u, v : [a, b] \to [0, +\infty)$ *be two (weight) functions. For any* $x \in [a, b]$*, we set*

$$w(x) = u(x) + v(x).$$

Let $f, g : [a, b] \to \mathbb{R}$ *be two functions such that* $u|f|^p$*,* $v|f|^p$*,* $u|g|^q$ *and* $v|g|^q$ *are integrable.*

(1) *Then we have*

$$\begin{aligned}
&\int_a^b w(x)|f(x)g(x)|dx \\
&\leq \left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q} \\
&+ \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q}.
\end{aligned}$$

(2) *Then we have*

$$\begin{aligned}
&\left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q} \\
&+ \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q} \\
&\leq \left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b w(x)|g(x)|^q dx\right)^{1/q}.
\end{aligned}$$

Proof. We will prove items (1) and (2) in turn.

(1) Using the definition of w and $1/p+1/q=1$, we can write

$$\int_a^b w(x)|f(x)g(x)|dx = \int_a^b (u(x)+v(x))|f(x)g(x)|dx$$
$$= \int_a^b u(x)|f(x)g(x)|dx + \int_a^b v(x)|f(x)g(x)|dx$$
$$= \int_a^b u(x)^{1/p}|f(x)|u(x)^{1/q}|g(x)|dx + \int_a^b v(x)^{1/p}|f(x)|v(x)^{1/q}|g(x)|dx.$$

Applying the Hölder integral inequality twice, we get

$$\int_a^b u(x)^{1/p}|f(x)|u(x)^{1/q}|g(x)|dx + \int_a^b v(x)^{1/p}|f(x)|v(x)^{1/q}|g(x)|dx$$
$$\leq \left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q}$$
$$+ \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q}.$$

Combining the above inequalities, we obtain

$$\int_a^b w(x)|f(x)g(x)|dx$$
$$\leq \left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q}$$
$$+ \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q}.$$

The desired inequality is established.

(2) Let us set

$$S = \left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b w(x)|g(x)|^q dx\right)^{1/q}.$$

Then we have

$$\frac{1}{S}\left\{\left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q}\right.$$
$$\left. + \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p}\left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q}\right\}$$

$$= \left(\frac{\int_a^b u(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right)^{1/p} \left(\frac{\int_a^b u(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)^{1/q}$$
$$+ \left(\frac{\int_a^b v(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right)^{1/p} \left(\frac{\int_a^b v(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)^{1/q}.$$

Applying the Young inequality, for any $m, n > 0$ and $\theta \in [0,1]$, $m^\theta n^{1-\theta} \leq \theta m + (1-\theta)n$, twice with $\theta = 1/p$ and appropriate m and n, and using the definition of w and $1/p + 1/q = 1$, we get

$$\left(\frac{\int_a^b u(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right)^{1/p} \left(\frac{\int_a^b u(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)^{1/q}$$
$$+ \left(\frac{\int_a^b v(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right)^{1/p} \left(\frac{\int_a^b v(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)^{1/q}$$
$$\leq \frac{1}{p}\left(\frac{\int_a^b u(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right) + \frac{1}{q}\left(\frac{\int_a^b u(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)$$
$$+ \frac{1}{p}\left(\frac{\int_a^b v(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right) + \frac{1}{q}\left(\frac{\int_a^b v(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)$$
$$= \frac{1}{p}\left(\frac{\int_a^b (u(x)+v(x))|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right) + \frac{1}{q}\left(\frac{\int_a^b (u(x)+v(x))|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)$$
$$= \frac{1}{p}\left(\frac{\int_a^b w(x)|f(x)|^p dx}{\int_a^b w(x)|f(x)|^p dx}\right) + \frac{1}{q}\left(\frac{\int_a^b w(x)|g(x)|^q dx}{\int_a^b w(x)|g(x)|^q dx}\right)$$
$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Combining the above inequalities, we obtain

$$\frac{1}{S}\left\{\left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q} + \right.$$
$$\left.\left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q}\right\}$$
$$\leq 1,$$

such that, by the definition of S,

$$\left(\int_a^b u(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b u(x)|g(x)|^q dx\right)^{1/q} + \left(\int_a^b v(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b v(x)|g(x)|^q dx\right)^{1/q} \leq \left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b w(x)|g(x)|^q dx\right)^{1/q}.$$

The desired inequality is established.

This completes the proof. □

If we take $u(x) = b - x$ and $v(x) = x - a$, so that $w(x) = b - a$, then Theorem 2.1 reduces to [4, Theorem 2.1]. If we assume that $w(x) = 1$, then Theorem 2.1 reduces to [4, Theorem 2.2]. Other choices of the weight functions lead to new Hölder-type integral inequalities.

3. Second theorem

Building on the work of [5, Theorem 2.1] and following the spirit of Theorem 2.1, the theorem below presents a generalized Hölder-type integral inequality involving two weight functions.

Theorem 3.1. *Let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *with* $b > a$. *Let* $p > 1$ *and* $q = p/(p-1)$. *Let* $u, v : [a, b] \to [0, +\infty)$ *be two (weight) functions. For any* $x \in [a, b]$, *we set*

$$w(x) = u(x) + v(x).$$

Let $f, g : [a, b] \to \mathbb{R}$ *be two functions such that* $u|f|$, $v|f|$, $uf|g|^q$ *and* $vf|g|^q$ *are integrable.*

(1) *Then we have*

$$\int_a^b w(x)|f(x)g(x)|dx \leq \left(\int_a^b u(x)|f(x)|dx\right)^{1/p} \left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q} + \left(\int_a^b v(x)|f(x)|dx\right)^{1/p} \left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}.$$

(2) *Then we have*

$$\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}$$
$$+\left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}$$
$$\leq\left(\int_a^b w(x)|f(x)|dx\right)^{1/p}\left(\int_a^b w(x)|f(x)||g(x)|^q dx\right)^{1/q}.$$

Proof. We will prove items (1) and (2) in turn.

(1) Using the definition of w and $1/p+1/q=1$, we can write

$$\int_a^b w(x)|f(x)g(x)|dx=\int_a^b (u(x)+v(x))|f(x)g(x)|dx$$
$$=\int_a^b u(x)|f(x)g(x)|dx+\int_a^b v(x)|f(x)g(x)|dx$$
$$=\int_a^b u(x)^{1/p}|f(x)|^{1/p}u(x)^{1/q}|f(x)|^{1/q}|g(x)|dx$$
$$+\int_a^b v(x)^{1/p}|f(x)|^{1/p}v(x)^{1/q}|f(x)|^{1/q}|g(x)|dx.$$

Applying the Hölder integral inequality twice, we get

$$\int_a^b u(x)^{1/p}|f(x)|^{1/p}u(x)^{1/q}|f(x)|^{1/q}|g(x)|dx$$
$$+\int_a^b v(x)^{1/p}|f(x)|^{1/p}v(x)^{1/q}|f(x)|^{1/q}|g(x)|dx$$
$$\leq\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}$$
$$+\left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}.$$

Combining the above inequalities, we obtain

$$\int_a^b w(x)|f(x)g(x)|dx$$
$$\leq\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}$$

$$+\left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}.$$

The desired inequality is established.

(2) Let us set

$$T=\left(\int_a^b w(x)|f(x)|dx\right)^{1/p}\left(\int_a^b w(x)|f(x)||g(x)|^q dx\right)^{1/q}.$$

Then we have

$$\frac{1}{T}\left\{\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}\right.$$
$$\left.+\left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}\right\}$$
$$=\left(\frac{\int_a^b u(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)^{1/p}\left(\frac{\int_a^b u(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)^{1/q}$$
$$+\left(\frac{\int_a^b v(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)^{1/p}\left(\frac{\int_a^b v(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)^{1/q}.$$

Applying the Young inequality, for any $m,n>0$ and $\theta\in[0,1]$, $m^\theta n^{1-\theta}\leq \theta m+(1-\theta)n$, twice with $\theta=1/p$ and appropriate m and n, and using the definition of w and $1/p+1/q=1$, we get

$$\left(\frac{\int_a^b u(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)^{1/p}\left(\frac{\int_a^b u(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)^{1/q}$$
$$+\left(\frac{\int_a^b v(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)^{1/p}\left(\frac{\int_a^b v(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)^{1/q}$$
$$\leq\frac{1}{p}\left(\frac{\int_a^b u(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)+\frac{1}{q}\left(\frac{\int_a^b u(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)$$
$$+\frac{1}{p}\left(\frac{\int_a^b v(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right)+\frac{1}{q}\left(\frac{\int_a^b v(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)$$

$$
\begin{aligned}
&= \frac{1}{p}\left(\frac{\int_a^b (u(x)+v(x))|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right) + \frac{1}{q}\left(\frac{\int_a^b (u(x)+v(x))|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)\\
&= \frac{1}{p}\left(\frac{\int_a^b w(x)|f(x)|dx}{\int_a^b w(x)|f(x)|dx}\right) + \frac{1}{q}\left(\frac{\int_a^b w(x)|f(x)||g(x)|^q dx}{\int_a^b w(x)|f(x)||g(x)|^q dx}\right)\\
&= \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{T}\left\{\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}\right.\\
&\left. + \left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}\right\}\\
&\le 1,
\end{aligned}
$$

so that, by the definition of T,

$$
\begin{aligned}
&\left(\int_a^b u(x)|f(x)|dx\right)^{1/p}\left(\int_a^b u(x)|f(x)||g(x)|^q dx\right)^{1/q}\\
&+ \left(\int_a^b v(x)|f(x)|dx\right)^{1/p}\left(\int_a^b v(x)|f(x)||g(x)|^q dx\right)^{1/q}\\
&\le \left(\int_a^b w(x)|f(x)|dx\right)^{1/p}\left(\int_a^b w(x)|f(x)||g(x)|^q dx\right)^{1/q}.
\end{aligned}
$$

The desired inequality is established.

This completes the proof. □

If we take $u(x) = b - x$ and $v(x) = x - a$, so that $w(x) = b - a$, then Theorem 2.1 reduces to [5, Theorem 2.1]. Other choices of the weight functions lead to new Hölder-type integral inequalities.

4. Conclusion

In this article, we have established two theorems that extend [4, Theorem 2.1] and [5, Theorem 2.1] by introducing intermediate weight functions whose sum need not equal one. The results, supported by detailed proofs, demonstrate a broader framework that generalizes the classical inequalities and adapts to a variety of settings. Further extensions to more complex weight structures or higher-dimensional domains could be investigated in future studies, potentially revealing new applications in analysis and related fields.

Conflict of interest

The authors declare no competing interests.

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