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EXTENSION OF SOLUTIONS OF THE BOUNDARY VALUE PROBLEM FOR A FIFTEEN VELOCITY MODEL IN THE HALF SPACE

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ABSTRACT. We extend in a half space the stationary solutions for the fifteen velocity model proven in a rectangle. We show that at infinity, these solutions tend towards a maxwellian state.

1. INTRODUCTION

In the kinetic theory of gases, the discretization of velocities, which consists of considering that the velocities of the particles belong to a finite set of vectors, is an already old idea. In 1960, Gross [9] underlined the interest of the discretization of velocities which makes it possible to replace Boltzmann's integro-differential equation by a system of non-linear and coupled partial differential equation. Simple gas models with discrete velocity distribution are then described and used to solve gas dynamics problems in which the Boltzmann equation must be used.The mathematical aspect of the study of discrete velocity models concern the proof of the global existence and uniqueness of the solutions of initial boundary value problem associated to the kinetic equations. In [8], the author proved the existence of stationary solutions of a boundary value problem modeling the one-dimensional flows of gas in half space. The situation is quite different when it

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comes to proving the existence of solutions to a boundary value problem modeling a two-dimensional flow of a gas in half space. The objective of this article is to extend in half space the stationary solutions for the fifteen velocity model whose existence is proven in rectangle in [1].

The paper is organized as follow. in section 2 we state the boundary value problem for which the existence of solutions is proved in section 3.

2. Statement of the problem

The steady flow of a gas in the half-space is a problem of gas dynamics, the modelling of which can lead to boundary value problems considered in this article. Therefore, the fifteen-velocity discrete model has been chosen to solve this problem. The velocity components of the fifteen-velocity discrete model that we will use to solve the problem are expressed in the $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ frame: $\vec{u}_1 = c(0,0,0), \vec{u}_2 = c(1,0,0), \vec{u}_3 = c(0,1,0), \vec{u}_4 = c(0,0,1), \vec{u}_{i+3} = -\vec{u}_i, i = 2,3,4, \vec{u}_8 = c(1,0,1), \vec{u}_9 = c(1,0,-1), \vec{u}_{12} = c(0,1,1), \vec{u}_{13} = c(0,1,-1), \vec{u}_{j+2} = -\vec{u}_j, i = 8,9,12,13.$ We denote by $N_i(t,x,y,z)$ the number density of particles of velocity $\vec{u}_i$ in point $M_i(t,x,y,z)$ at time t. The N_i are continuous functions of t, x, y and z. The kinetic equations of the model are given in the Appendix. We assume that the flow is bidirectional and depends only on the variables x and y. This assumption retained together with the steadiness of the flow implies the vanishing of the partial derivatives with respect to t and z. Furthermore we assume that the half-space is:

$$\{(x,y) \in \mathbb{R}^2 / x \geq 0\} = [0, +\infty[\times]-\infty, +\infty[.$$

The system of equations describing the evolution of the microscopic densities $N_i(x,y)$, $i \in \Lambda$, $(x,y) \in [0,+\infty[\times]-\infty,+\infty[$ are given by equations

$$\begin{aligned}
0 &= \sqrt{2}cs[(N_2+N_3+N_5+N_6)(N_4+N_7) - (N_8+N_9+N_{10}+N_{11}+N_{12}+N_{13})N_1] \\
&\quad - \sqrt{2}cs[(N_{14}+N_{15})N_1] \\
c\frac{\partial N_2}{\partial x'} &= \sqrt{3}cs\left[(N_3+N_6)(N_8+N_9) - N_2(N_{12}+N_{13}+N_{14}+N_{15})\right] \\
&\quad + \sqrt{5}cs\left[N_3(N_{14}+N_{15}) + N_5(N_8+N_9) + N_6(N_{12}+N_{13}) - 3N_1(N_{10}+N_{11})\right] \\
&\quad + \sqrt{2}cs\left[N_1(N_8+N_9) - N_2(N_4+N_7)\right] + 2cs\left(N_3N_6 + N_4N_7 - 2N_2N_5\right) \\
c\frac{\partial N_3}{\partial y'} &= \sqrt{3}cs\left[(N_3+N_6)(N_{12}+N_{13}) - N_3(N_8+N_9+N_{10}+N_{11})\right] \\
&\quad + \sqrt{5}cs\left[N_2(N_{10}+N_{11}) + N_5(N_8+N_9) + N_6(N_{12}+N_{13}) - 3N_3(N_{14}+N_{15})\right]
\end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&+\sqrt{2}cs\left[N_1\left(N_{12}+N_{13}\right)-N_3\left(N_4+N_7\right)\right]+2cs\left(N_2N_5+N_4N_7-N_3N_6\right)\\
0 &= \sqrt{2}cs[N_1(N_8+N_{11}+N_{12}+N_{15})-N_4(N_2+N_3+N_5+N_6)]+2cs(N_2N_5\\
&+N_3N_6-2N_4N_7)+\sqrt{5}cs[N_7(N_8+N_{11}+N_{12}+N_{15})-N_4(N_9+N_{10}+N_{13}+N_{14})]\\
-c\frac{\partial N_5}{\partial x'} &= \sqrt{3}cs\left[\left(N_3+N_6\right)\left(N_{10}+N_{11}\right)-N_5\left(N_{12}+N_{13}+N_{14}+N_{15}\right)\right]\\
&+\sqrt{5}cs\left[N_2\left(N_{10}+N_{11}\right)+N_3\left(N_{14}+N_{15}\right)+N_6\left(N_{12}+N_{13}\right)-3N_5\left(N_8+N_9\right)\right]\\
&+\sqrt{2}cs\left[N_1\left(N_{10}+N_{11}\right)-N_5\left(N_4+N_7\right)\right]+2cs\left(N_3N_6+N_4N_7-2N_2N_5\right)\\
-c\frac{\partial N_6}{\partial y'} &= \sqrt{3}cs\left[\left(N_2+N_5\right)\left(N_{14}+N_{15}\right)-N_6\left(N_8+N_9+N_{10}+N_{11}\right)\right]\\
&+\sqrt{5}cs\left[N_2\left(N_{10}+N_{11}\right)+N_3\left(N_{14}+N_{15}\right)+N_5\left(N_8+N_9\right)-3N_6\left(N_{12}+N_{13}\right)\right]\\
&+\sqrt{2}cs\left[N_1\left(N_{14}+N_{15}\right)-N_6\left(N_4+N_7\right)\right]+2cs\left(N_2N_5+N_4N_7-2N_3N_6\right)\\
0 &= \sqrt{2}cs[N_1(N_9+N_{10}+N_{13}+N_{14})-N_7(N_2+N_3+N_5+N_6)]+2cs(N_2N_5\\
&+N_3N_6-2N_4N_7)+\sqrt{5}cs[N_4(N_9+N_{10}+N_{13}+N_{14})-N_7(N_8+N_{11}+N_{12}+N_{15})]\\
c\frac{\partial N_8}{\partial x'} &= 2cs(N_{12}N_{15}-N_8N_{11})+\sqrt{5}cs(N_2N_{11}+N_3N_{15}+N_4N_9+N_6N_{12}\\
&-3N_5N_8-N_7N_8)+\sqrt{3}cs[N_2(N_{12}+N_{15})-N_8(N_3+N_6)]+\sqrt{6}cs[N_9(N_{12}+N_{15})-N_8N_{13}]\\
&-\sqrt{6}csN_8N_{14}\\
&+2\sqrt{2}cs(N_9N_{11}+N_{12}N_{14}+N_{13}N_{15}-3N_8N_{10})+\sqrt{2}cs(N_2N_4-N_1N_8)\\
c\frac{\partial N_9}{\partial x'} &= \sqrt{3}cs\left[N_2\left(N_{13}+N_{14}\right)-N_8\left(N_3+N_6\right)\right]+2cs\left(N_{13}N_{14}-N_9N_{10}\right)\\
&+\sqrt{2}cs\left(N_2N_4-N_1N_8\right)+\sqrt{5}cs\left(N_2N_{10}+N_3N_{14}+N_6N_{13}+N_7N_8-N_4N_9-3N_5N_9\right)\\
&+\sqrt{6}cs\left[N_8\left(N_{13}+N_{14}\right)-N_9\left(N_{12}+N_{15}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_{12}N_{14}+N_{13}N_{15}-3N_9N_{11}\right)\\
-c\frac{\partial N_{10}}{\partial x'} &= \sqrt{3}cs\left[N_5\left(N_{13}+N_{14}\right)-N_{10}\left(N_3+N_6\right)\right]+2cs\left(N_{13}N_{14}-N_9N_{10}\right)\\
&+\sqrt{2}cs\left(N_5N_7-N_1N_{10}\right)+\sqrt{5}cs\left(N_3N_{14}+N_5N_9+N_6N_{13}+N_7N_{11}-N_4N_{10}-3N_2N_{10}\right)\\
&+\sqrt{6}cs\left[N_{11}\left(N_{13}+N_{14}\right)-N_{10}\left(N_{12}+N_{15}\right)\right]+2\sqrt{2}cs\left(N_9N_{11}+N_{12}N_{14}+N_{13}N_{15}-3N_8N_{10}\right)\\
-c\frac{\partial N_{11}}{\partial x'} &= \sqrt{3}cs\left[N_5\left(N_{12}+N_{15}\right)-N_{11}\left(N_3+N_6\right)\right]+2cs\left(N_{12}N_{15}-N_8N_{11}\right)\\
&+\sqrt{2}cs\left(N_4N_5-N_1N_{11}\right)+\sqrt{5}cs\left(N_3N_{13}+N_5N_8+N_6N_{12}+N_4N_{10}-N_7N_{11}-3N_2N_{11}\right)\\
&+\sqrt{6}cs\left[N_{10}\left(N_{12}+N_{15}\right)-N_{11}\left(N_{13}+N_{14}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_{12}N_{14}+N_{13}N_{15}-3N_9N_{11}\right)\\
c\frac{\partial N_{12}}{\partial y'} &= \sqrt{3}cs\left[N_3\left(N_8+N_{11}\right)-N_{12}\left(N_2+N_5\right)\right]+2cs\left(N_8N_{11}-N_{12}N_{15}\right)\\
&+\sqrt{2}cs\left(N_3N_4-N_1N_{12}\right)+\sqrt{5}cs\left(N_2N_{11}+N_3N_{15}+N_4N_{13}+N_5N_8-N_7N_{12}-3N_6N_{12}\right)\\
&+\sqrt{6}cs\left[N_{13}\left(N_8+N_{11}\right)-N_{12}\left(N_9+N_{10}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_9N_{11}+N_{13}N_{15}-3N_{12}N_{14}\right)\\
c\frac{\partial N_{13}}{\partial y'} &= \sqrt{3}cs\left[N_3\left(N_9+N_{10}\right)-N_{31}\left(N_2+N_5\right)\right]+2cs\left(N_9N_{10}-N_{13}N_{14}\right)\\
&+\sqrt{2}cs\left(N_3N_7-N_1N_{13}\right)+\sqrt{5}cs\left(N_2N_{110}+N_3N_{14}+N_5N_9+N_7N_{12}-N_4N_{13}-3N_6N_{13}\right)\\
&+\sqrt{6}cs\left[N_{12}\left(N_9+N_{10}\right)-N_{13}\left(N_8+N_{11}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_9N_{11}+N_{12}N_{14}-3N_{13}N_{15}\right)
\end{aligned}
$$

$$
\begin{aligned}
-c\frac{\partial N_{14}}{\partial y'} &= \sqrt{3}cs\left[N_6\left(N_9+N_{10}\right)-N_{14}\left(N_2+N_5\right)\right]+2cs\left(N_9N_{10}-N_{13}N_{14}\right)\\
&+\sqrt{2}cs\left(N_6N_7-N_1N_{14}\right)+\sqrt{5}cs\left(N_2N_{10}+N_5N_9+N_6N_{13}+N_7N_{15}-N_4N_{14}-3N_3N_{14}\right)\\
&+\sqrt{6}cs\left[N_{15}\left(N_9+N_{10}\right)-N_{14}\left(N_8+N_{11}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_9N_{11}+N_{13}N_{15}-3N_{12}N_{14}\right)\\
-c\frac{\partial N_{15}}{\partial y'} &= \sqrt{3}cs\left[N_6\left(N_8+N_{11}\right)-N_{15}\left(N_2+N_5\right)\right]+2cs\left(N_8N_{11}-N_{12}N_{15}\right)\\
&+\sqrt{2}cs\left(N_4N_6-N_1N_{15}\right)+\sqrt{5}cs\left(N_2N_{11}+N_5N_8+N_6N_{12}+N_4N_{14}-N_7N_{15}-3N_3N_{15}\right)\\
&+\sqrt{6}cs\left[N_{14}\left(N_8+N_{11}\right)-N_{15}\left(N_9+N_{10}\right)\right]+2\sqrt{2}cs\left(N_8N_{10}+N_9N_{11}+N_{12}N_{14}-3N_{13}N_{15}\right)
\end{aligned}
$$

Here Λ denotes the set of the indices of the velocities of the discrete model. The system (2.1) is completed by boundary conditions. On the interface $x = 0$, boundary conditions are given for the densities of molecules leaving the gas toward the interface. At infinity $(-\infty \ or \ +\infty)$, boundary conditions are given for all densities. In summary, the boundary conditions are of the form:

$$
(2.2)\qquad \begin{cases}
N_k\left(0,y\right) &= \phi_k^0\left(y\right),\ k\in\Lambda_1\\
\lim\limits_{x\to+\infty} N_k\left(x,y\right) &= \phi_k^{+\infty}\left(y\right),\ k\in\Lambda\\
\lim\limits_{y\to-\infty} N_k\left(x,y\right) &= \psi_k^{-\infty}\left(x\right),\ k\in\Lambda\\
\lim\limits_{y\to+\infty} N_k\left(x,y\right) &= \psi_k^{+\infty}\left(x\right),\ k\in\Lambda
\end{cases}
$$

where $\Lambda = \{1,2,3\ldots,15\}$, $\Lambda_1 = \{2,8,9\}$. The boundary value problem (2.1)-(2.2) comes from the use of the fifteen-velocity model to describe the steady, two-dimensional flow of a gas in the half-space $[0,+\infty[\ \times\ ]-\infty,+\infty[$. The objective of this article is to prove the existence of solutions to the boundary value problem (2.1)-(2.2).

Theorem 2.1. *If the boundary data at infinity $\psi_k^{\pm\infty}$ and $\phi_k^{+\infty}$ are the densities of a Maxwellian state then the boundary value problem* (2.1)-(2.2) *has a solution.*

3. Existence of solutions of (2.1)-(2.2)

To prove Theorem (2.1) we first show that the boundary value problem (2.1)-(2.2) in the strip $[0;a]\times]-\infty;+\infty[$, i.e the system (2.1) with the boundary conditions

$$\tag{3.1} \begin{cases} N_k\left(0,y\right) & = & f_k^0\left(y\right), k\in\{2,8,9\} \\ N_k\left(a,y\right) & = & f_k^a\left(y\right), k\in\{5,10,11\} \\ \lim\limits_{y\to-\infty} N_k\left(x,y\right) & = & g_k^{-\infty}\left(x\right), k\in\Lambda \\ \lim\limits_{y\to+\infty} N_k\left(x,y\right) & = & g_k^{+\infty}\left(x\right), k\in\Lambda \end{cases}$$

has continuous and bounded solutions independently of a. To do this, we rely on the fact that it is shown in [1] that the boundary value problem (2.1)-(2.2) in the rectangle $[0,a]\times[-b;b]$ has continuous and bounded solution independently of a and b on $[0,a]\times[-b;b]$, and we show that the sequence of solutions of the boundary value problem in the rectangle, obtained for an increasing sequence $(b_l)_{l\in\mathbb{N}}$ of strictly positive real numbers converges to a solution of (2.1)-(3.1). We then show that the sequence of solutions of (2.1)-(3.1) obtained for an increasing sequence $(a_l)_{l\in\mathbb{N}}$ of strictly positive real numbers converges towards a solution of (2.1)-(2.2).

The boundary value problem (2.1)-(2.2) in a rectangle $[0,a]\times[-b,b]$ is given by the system (2.1) with the boundary condition

$$\tag{3.2} \begin{cases} N_2(0,y) & = & \phi_2^0(y) \\ N_8(0,y) & = & \phi_8^0(y) \\ N_9(0,y) & = & \phi_9^0(y) \\ N_5(a,y) & = & \phi_5^a(y) \\ N_{10}(a,y) & = & \phi_{10}^a(y) \\ N_{11}(a,y) & = & \phi_{11}^a(y) \\ N_3(x,-b) & = & \psi_3^{-b}(x) \\ N_{12}(x,-b) & = & \psi_{12}^{-b}(x) \\ N_{13}(x,-b) & = & \psi_{13}^{-b}(x) \\ N_6(x,b) & = & \psi_6^b(x) \\ N_{14}(x,b) & = & \psi_{14}^b(x) \\ N_{15}(x,b) & = & \psi_{15}^b(x) \end{cases}$$

and it was shown in [1] that the boundary value problem (2.1)-(3.2) has uniformly continuous and independently bounded solutions of a and b on $[0,a]\times[-b,b]$ and their set is also equicontinuous. Thus let $(b_l)_{l\in\mathbb{N}}$ be an increasing sequence of positive real numbers tending toward $+\infty$, the solutions N^l of (2.1)-(3.2) on $[0,a]\times[-b_l,b_l]$ are uniformly continuous and bounded independently of a and b_l and their set is also equicontinuous for all $l\in\mathbb{N}$. We extend N^l to $[0,a]\times]-\infty,+\infty[$ by the function $\widetilde{N}^l$ defined by:

$$\widetilde{N}^l(x,y)=\begin{cases} N^l(x,y) & ,(x,y)\in[0;a]\times[-b_l;b_l] \\ N^l(x,b_l) & ,(x,y)\in[0;a]\times\mathbb{R}, y>b_l \\ N^l(x,-b_l) & ,(x,y)\in[0;a]\times\mathbb{R}, y<-b_l \end{cases}.$$

The sequence $\left(\widetilde{N}^l\right)_{l\in\mathbb{N}}$ is clearly a sequence of uniformly continuous and bounded functions satisfying (2.1)-(3.1).

Proposition 3.1. *There exist a subsequence of* $\left(\widetilde{N}^l\right)_{l\in\mathbb{N}}$ *which simply converges on* $[0;a]\times]-\infty;+\infty[$ *and uniformly on any bounded interval from* $[0;a]\times]-\infty;+\infty[$ *towards a uniformly continuous and bounded function* N

Proof. The function $\widetilde{N}^l$ are bounded on $[0,a]\times]-\infty;+\infty[$ therefore $\forall k\in\Lambda,\forall(x,y)\in[0;a]\times]-\infty;+\infty[$, the sequence $\left(\widetilde{N}^l_k(x,y)\right)_{l\in\mathbb{N}}$ is a bounded sequence of positive real numbers. We can therefore extract a convergent subsequence $\left(\widetilde{N}^{l_k}_k(x,y)\right)_{l_k\in\mathbb{N}}$.
Let $\widetilde{N}_k(x,y)$ be the limit of this subsequence. $\forall k\in\Lambda$, we define on $[0,a]\times]-\infty,+\infty[$ the function $N=(N_k)_{k\in\Lambda}$ by $N_k(x,y)=\widetilde{N}_k(x,y)$. The sequence of functions $\left(\widetilde{N}^l\right)_{l\in\mathbb{N}}$ simply converges to N on $[0,a]\times]-\infty,+\infty[$.
Let $(x,y),(x',y')\in[0,a]\times]-\infty,+\infty[$. By adding the quantities $\widetilde{N}^l_k(x,y)$ and $\widetilde{N}^l_k(x',y')$ and subtracting them inside the absolute value $|N_k(x,y)-N_k(x',y')|$ we obtain:

$$\begin{aligned} & |N_k(x,y)-N_k(x',y')| \\ = & \left|N_k(x,y)-N_k(x',y')+\widetilde{N}^l_k(x,y)-\widetilde{N}^l_k(x,y)+\widetilde{N}^l_k(x',y')-\widetilde{N}^l_k(x',y')\right| \\ = & \left|\left(N_k(x,y)-\widetilde{N}^l_k(x,y)\right)+\left(\widetilde{N}^l_k(x',y')-N_k(x',y')\right)+\left(\widetilde{N}^l_k(x,y)-\widetilde{N}^l_k(x',y')\right)\right| \\ \leq & \left|\widetilde{N}^l_k(x,y)-N_k(x,y)\right|+\left|\widetilde{N}^l_k(x',y')-N_k(x',y')\right|+\left|\widetilde{N}^l_k(x',y')-\widetilde{N}^l_k(x,y)\right| \end{aligned}$$

As $(\widetilde{N}^l)_{l\in\mathbb{N}}$ simply converges to N on $[0;a]\times]-\infty;+\infty[$ we have:

$$\begin{aligned} &\forall(x,y)\in[0,a]\times]-\infty,+\infty[,\forall\varepsilon>0,\exists l_0\in\mathbb{N},\forall l\in\mathbb{N},l>l_0 \\ \Longrightarrow\ &\left|\widetilde{N}_k^l(x,y)-N_k(x,y)\right|<\frac{\varepsilon}{3} \end{aligned} \tag{3.3}$$

Moreover $\widetilde{N}^l$ is uniformly continuous $\forall l\in\mathbb{N}$ and $\forall k\in\Lambda$ therefore

$$\begin{aligned} &\forall\varepsilon>0,\exists\alpha>0,\forall(x,y)\in[0,a]\times]-\infty,+\infty[,\forall(x',y')\in[0,a]\times]-\infty,+\infty[, \\ &|x-x'|<\alpha,|y-y'|<\alpha \\ \Longrightarrow\ &\left|\widetilde{N}_k^l(x,y)-\widetilde{N}_k^l(x',y')\right|<\frac{\varepsilon}{3} \end{aligned} \tag{3.4}$$

We deduce from (3.3) and (3.4) that:

$$\begin{aligned} &\forall\varepsilon>0,\exists\alpha>0,\exists l_0\in\mathbb{N},\forall(x,y),(x',y')\in[0,a]\times]-\infty,+\infty[,\forall l\in\mathbb{N}, \\ &l>l_0,|x-x'|<\alpha,|y-y'|<\alpha \\ \Longrightarrow\ &|N_k\,(x,y)-N_k\,(x',y')|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \end{aligned} \tag{3.5}$$

Hence, N is uniformly continuous.

It remains to show that $\left(\widetilde{N}^l\right)_{l\in\mathbb{N}}$ converges uniformly on any bounded closed interval from $[0,a]\times]-\infty,+\infty[$ to N to finish the proof. As on any closed bounded interval of $[0,a]\times]-\infty,+\infty[$, $\widetilde{N}^l=N^l$, this amounts to showing that $(N^l)_{l\in\mathbb{N}}$ converges uniformly towards N.

Let $(x,y),(x',y')\in[0,a]\times[-b_l,b_l],\forall l\in\mathbb{N}$,

$$\begin{aligned} &\left|N_k^l(x,y)-N_k(x,y)\right| \\ =\ &\left|N_k^l(x,y)+N_k^l(x',y')-N_k^l(x',y')-N_k(x,y)+N_k(x',y')-N_k(x',y')\right| \\ \leq\ &\left|N_k^l(x,y)-N_k^l(x',y')\right|+\left|N_k^l(x',y')-N_k(x',y')\right|+\left|N_k(x',y')-N_k(x,y)\right|. \end{aligned}$$

As N^l is uniformly continuous then:

$$\begin{aligned} &\forall\varepsilon>0,\exists\alpha>0,\forall(x,y)\in[0,a]\times[-b_l,b_l],\forall(x',y')\in[0,a]\times]-b_l,b_l[, \\ &|x-x'|<\alpha,|y-y'|<\alpha\Longrightarrow\left|N_k^l(x,y)-N_k^l(x',y')\right|<\frac{\varepsilon}{3}. \end{aligned} \tag{3.6}$$

As N^l simply converges to N on $[0,a]\times[-b_l,b_l]$ then:

$$\begin{aligned} &\forall(x',y')\in[0,a]\times[-b_l,b_l],\forall\varepsilon>0,\exists l_0\in\mathbb{N},\forall l\in\mathbb{N},l>l_0 \\ \Longrightarrow\ &\left|N_k^l(x',y')-N_k(x',y')\right|<\frac{\varepsilon}{3}. \end{aligned} \tag{3.7}$$

As N is uniformly continuous then:

$$\begin{aligned}&\forall\varepsilon>0,\exists\alpha>0,\forall(x,y),(x',y')\in[0,a]\times[-b_l,b_l],\\ &|x-x'|<\alpha,|y-y'|<\alpha\Longrightarrow|N_k(x',y')-N_k(x,y)|<\frac{\varepsilon}{3}.\end{aligned}\tag{3.8}$$

From (3.6), (3.7) and (3.8) we deduce that:

$$\begin{aligned}&\forall\varepsilon>0,\exists\alpha>0,\exists l_0\in\mathbb{N},\forall(x,y),(x',y')\in[0,a]\times]-b_l,b_l[,\forall l\in\mathbb{N},\\ &l>l_0,|x-x'|<\alpha,|y-y'|<\alpha\\ \Longrightarrow\ &\left|N_k^l\left(x,y\right)-N_k\left(x,y\right)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}\tag{3.9}$$

So, $(\widetilde{N}^l)_{l\in\mathbb{N}}$ uniformly converges to N. □

Proposition 3.2. *For any increasing sequence $(y_k)_{k\in\mathbb{N}}$ of positive real numbers tending towards $+\infty$ the sequence $(N\left(x,y_k\right))_{k\in\mathbb{N}}$ tends to a Maxwellian state when k tends to $+\infty$ $\forall x\in[0,a]$.*

Proof. We set $\overrightarrow{u}_k = u_k^\alpha\overrightarrow{e}_\alpha, h_\alpha\left(x,y\right) = \sum_{k=1}^{15}N_k\left(x,y\right)\ln N_k\left(x,y\right)u_k^\alpha$ and $\overrightarrow{H} = \sum_{\alpha=1}^{2}h_\alpha\overrightarrow{e}_\alpha$. h_α is a continuous function on $[0;a]\times]-\infty,+\infty[$ for all α. We have:

$$\begin{aligned}&\overrightarrow{\nabla}\cdot\overrightarrow{H}=\frac{\partial h_1}{\partial x}+\frac{\partial h_2}{\partial y}\\ =\ &c\left(1+\ln N_2\right)\frac{\partial N_2}{\partial x}-c\left(1+\ln N_5\right)\frac{\partial N_5}{\partial x}+c\left(1+\ln N_8\right)\frac{\partial N_8}{\partial x}+c\left(1+\ln N_9\right)\frac{\partial N_9}{\partial x}\\ -\ &c\left(1+\ln N_{10}\right)\frac{\partial N_{10}}{\partial x}-c\left(1+\ln N_{11}\right)\frac{\partial N_{11}}{\partial x}+c\left(1+\ln N_3\right)\frac{\partial N_3}{\partial y}\\ -\ &c\left(1+\ln N_6\right)\frac{\partial N_6}{\partial y}+c\left(1+\ln N_{12}\right)\frac{\partial N_{12}}{\partial y}+c\left(1+\ln N_{13}\right)\frac{\partial N_{13}}{\partial y}\\ -\ &c\left(1+\ln N_{14}\right)\frac{\partial N_{14}}{\partial y}-c\left(1+\ln N_{15}\right)\frac{\partial N_{15}}{\partial y}.\end{aligned}$$

By multiplying the first equation of (2.1) par $1+\ln N_1$, the second by $1+\ln N_2,\ldots,$ the k^{eme} by $1+\ln N_k$ and by making a member-by-member sum, we obtain:

$$\begin{aligned}&\overrightarrow{\nabla}\cdot\overrightarrow{H}\\ =\ &\sqrt{2}cs\ln\left(\frac{N_2N_4}{N_1N_8}\right)\left(1-\frac{N_2N_4}{N_1N_8}\right)N_1N_8+\sqrt{2}cs\ln\left(\frac{N_2N_7}{N_1N_9}\right)\left(1-\frac{N_2N_7}{N_1N_9}\right)N_1N_9\\ +\ &\sqrt{2}cs\ln\left(\frac{N_3N_4}{N_1N_{12}}\right)\left(1-\frac{N_3N_4}{N_1N_{12}}\right)N_1N_{12}+\sqrt{2}cs\ln\left(\frac{N_3N_7}{N_1N_{13}}\right)\left(1-\frac{N_3N_7}{N_1N_{13}}\right)N_1N_{13}\\ +\ &\sqrt{2}cs\ln\left(\frac{N_5N_7}{N_1N_{10}}\right)\left(1-\frac{N_5N_7}{N_1N_{10}}\right)N_1N_{10}+\sqrt{2}cs\ln\left(\frac{N_4N_5}{N_1N_{11}}\right)\left(1-\frac{N_4N_5}{N_1N_{11}}\right)N_1N_{11}\end{aligned}\tag{3.10}$$

$$
\begin{aligned}
&+ \sqrt{2}cs\ln\left(\frac{N_4N_6}{N_1N_{15}}\right)\left(1-\frac{N_4N_6}{N_1N_{15}}\right)N_1N_{15}+\sqrt{2}cs\ln\left(\frac{N_6N_7}{N_1N_{14}}\right)\left(1-\frac{N_6N_7}{N_1N_{14}}\right)N_1N_{14}\\
&+ 2cs\ln\left(\frac{N_2N_5}{N_3N_6}\right)\left(1-\frac{N_2N_5}{N_3N_6}\right)N_3N_6+2cs\ln\left(\frac{N_2N_5}{N_4N_7}\right)\left(1-\frac{N_2N_5}{N_4N_7}\right)N_4N_7\\
&+ 2cs\ln\left(\frac{N_3N_6}{N_4N_7}\right)\left(1-\frac{N_3N_6}{N_4N_7}\right)N_4N_7+2cs\ln\left(\frac{N_8N_{11}}{N_{12}N_{15}}\right)\left(1-\frac{N_8N_{11}}{N_{12}N_{15}}\right)N_{12}N_{15}\\
&+ 2cs\ln\left(\frac{N_9N_{10}}{N_{13}N_{14}}\right)\left(1-\frac{N_9N_{10}}{N_{13}N_{14}}\right)N_{13}N_{14}+\sqrt{3}cs\ln\left(\frac{N_2N_{12}}{N_3N_8}\right)\left(1-\frac{N_2N_{12}}{N_3N_8}\right)N_3N_8\\
&+ \sqrt{3}cs\ln\left(\frac{N_2N_{13}}{N_3N_9}\right)\left(1-\frac{N_2N_{13}}{N_3N_9}\right)N_3N_9+\sqrt{3}cs\ln\left(\frac{N_2N_{15}}{N_6N_8}\right)\left(1-\frac{N_2N_{15}}{N_6N_8}\right)N_6N_8\\
&+ \sqrt{3}cs\ln\left(\frac{N_2N_{14}}{N_6N_9}\right)\left(1-\frac{N_2N_{14}}{N_6N_9}\right)N_6N_9+\sqrt{3}cs\ln\left(\frac{N_5N_{12}}{N_3N_{11}}\right)\left(1-\frac{N_5N_{12}}{N_3N_{11}}\right)N_3N_{11}\\
&+ \sqrt{3}cs\ln\left(\frac{N_5N_{13}}{N_3N_{10}}\right)\left(1-\frac{N_5N_{13}}{N_3N_{10}}\right)N_3N_{10}+\sqrt{3}cs\ln\left(\frac{N_5N_{14}}{N_6N_{10}}\right)\left(1-\frac{N_5N_{14}}{N_6N_{10}}\right)N_6N_{10}\\
&+ \sqrt{3}cs\ln\left(\frac{N_5N_{15}}{N_6N_{11}}\right)\left(1-\frac{N_5N_{15}}{N_6N_{11}}\right)N_6N_{11}+\sqrt{5}cs\ln\left(\frac{N_2N_{10}}{N_3N_{14}}\right)\left(1-\frac{N_2N_{10}}{N_3N_{14}}\right)N_3N_{14}\\
&+ \sqrt{5}cs\ln\left(\frac{N_2N_{11}}{N_3N_{15}}\right)\left(1-\frac{N_2N_{11}}{N_3N_{15}}\right)N_3N_{15}+\sqrt{5}cs\ln\left(\frac{N_2N_{11}}{N_5N_8}\right)\left(1-\frac{N_2N_{11}}{N_5N_8}\right)N_5N_8\\
&+ \sqrt{5}cs\ln\left(\frac{N_2N_{10}}{N_5N_9}\right)\left(1-\frac{N_2N_{10}}{N_5N_9}\right)N_5N_9+\sqrt{5}cs\ln\left(\frac{N_2N_{11}}{N_6N_{12}}\right)\left(1-\frac{N_2N_{11}}{N_6N_{12}}\right)N_6N_{12}\\
&+ \sqrt{5}cs\ln\left(\frac{N_2N_{10}}{N_6N_{13}}\right)\left(1-\frac{N_2N_{10}}{N_6N_{13}}\right)N_6N_{13}+\sqrt{5}cs\ln\left(\frac{N_3N_{14}}{N_6N_{13}}\right)\left(1-\frac{N_3N_{14}}{N_6N_{13}}\right)N_6N_{13}\\
&+ \sqrt{5}cs\ln\left(\frac{N_3N_{14}}{N_5N_9}\right)\left(1-\frac{N_3N_{14}}{N_5N_9}\right)N_5N_9+\sqrt{5}cs\ln\left(\frac{N_3N_{15}}{N_5N_8}\right)\left(1-\frac{N_3N_{15}}{N_5N_8}\right)N_5N_8\\
&+ \sqrt{5}cs\ln\left(\frac{N_3N_{15}}{N_6N_{12}}\right)\left(1-\frac{N_3N_{15}}{N_6N_{12}}\right)N_6N_{12}+\sqrt{5}cs\ln\left(\frac{N_5N_8}{N_6N_{12}}\right)\left(1-\frac{N_5N_8}{N_6N_{12}}\right)N_6N_{12}\\
&+ \sqrt{5}cs\ln\left(\frac{N_5N_9}{N_6N_{13}}\right)\left(1-\frac{N_5N_9}{N_6N_{13}}\right)N_6N_{13}+\sqrt{5}cs\ln\left(\frac{N_7N_8}{N_4N_9}\right)\left(1-\frac{N_7N_8}{N_4N_9}\right)N_4N_9\\
&+ \sqrt{5}cs\ln\left(\frac{N_4N_{10}}{N_7N_{11}}\right)\left(1-\frac{N_4N_{10}}{N_7N_{11}}\right)N_7N_{11}+\sqrt{5}cs\ln\left(\frac{N_7N_{12}}{N_4N_{13}}\right)\left(1-\frac{N_7N_{12}}{N_4N_{13}}\right)N_4N_{13}\\
&+ \sqrt{5}cs\ln\left(\frac{N_4N_{14}}{N_7N_{15}}\right)\left(1-\frac{N_4N_{14}}{N_7N_{15}}\right)N_7N_{15}+\sqrt{6}cs\ln\left(\frac{N_8N_{13}}{N_9N_{12}}\right)\left(1-\frac{N_8N_{13}}{N_9N_{12}}\right)N_9N_{12}\\
&+ \sqrt{6}cs\ln\left(\frac{N_8N_{14}}{N_9N_{15}}\right)\left(1-\frac{N_8N_{14}}{N_9N_{15}}\right)N_9N_{15}+\sqrt{6}cs\ln\left(\frac{N_{10}N_{12}}{N_{11}N_{13}}\right)\left(1-\frac{N_{10}N_{12}}{N_{11}N_{13}}\right)N_{11}N_{13}\\
&+ \sqrt{6}cs\ln\left(\frac{N_{11}N_{14}}{N_{15}N_{10}}\right)\left(1-\frac{N_{11}N_{14}}{N_{15}N_{10}}\right)N_{15}N_{10}+2\sqrt{2}cs\ln\left(\frac{N_8N_{10}}{N_9N_{11}}\right)\left(1-\frac{N_8N_{10}}{N_9N_{11}}\right)N_9N_{11}\\
&+ 2\sqrt{2}cs\ln\left(\frac{N_8N_{10}}{N_{12}N_{14}}\right)\left(1-\frac{N_8N_{10}}{N_{12}N_{14}}\right)N_{12}N_{14}+2\sqrt{2}cs\ln\left(\frac{N_8N_{10}}{N_{13}N_{15}}\right)\left(1-\frac{N_8N_{10}}{N_{13}N_{15}}\right)N_{13}N_{15}\\
&+ 2\sqrt{2}cs\ln\left(\frac{N_9N_{11}}{N_{12}N_{14}}\right)\left(1-\frac{N_9N_{11}}{N_{12}N_{14}}\right)N_{12}N_{14}+2\sqrt{2}cs\ln\left(\frac{N_9N_{11}}{N_{13}N_{15}}\right)\left(1-\frac{N_9N_{11}}{N_{13}N_{15}}\right)N_{13}N_{15}\\
&+ 2\sqrt{2}cs\ln\left(\frac{N_{12}N_{14}}{N_{13}N_{15}}\right)\left(1-\frac{N_{12}N_{14}}{N_{13}N_{15}}\right)N_{13}N_{15}.
\end{aligned}
$$

The function $z \longmapsto (1-z)\ln z$ is negative or zero for $z > 0$ (zero for $z = 1$). As $\sqrt{2}cs$, $2cs$, $\sqrt{3}cs$, $\sqrt{5}cs$, $\sqrt{6}cs$, $2\sqrt{2}cs$ are positive as well as $N_k, k \in \{1, 2, 3, \cdots, 15\}$, $\forall (x, y) \in [0; a] \times]-\infty; +\infty[$, $\overrightarrow{\nabla} \cdot \overrightarrow{H} \leq 0$. We deduce that $\displaystyle\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dy \leq 0$ $\forall x \in [0; a]$. N is bounded on $[0; a] \times]-\infty; +\infty[$ as the limit of a sequence of functions bounded on $[0; a] \times]-\infty; +\infty[$ so $\displaystyle\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dy$ exists and is finite, negative or zero and is a function of x. Let $J_k = [y_k, y_{k+1}[$ and $\varphi(J_k) = -\int_{y_k}^{y_{k+1}} \overrightarrow{\nabla} \cdot \overrightarrow{H} dy = \varphi_k(x)$ a function.

As $\overrightarrow{\nabla}.\overrightarrow{H} \leq 0$, $\varphi(J_k)$ is a positive or zero function. Thus the sequence of function $(\varphi(J_k))_{k\in\mathbb{N}}$ is a sequence of positive functions(of positive functions of x) and the series of function of x, $\sum_k \varphi(J_k)$ is a series of positive functions. Let $S_k = \sum_{n=0}^{k} \varphi(J_n)(x)$ its partial sums. Let S be the function defined from $[0;a]$ to $\mathbb{R}^+$ which to $x \longmapsto S(x) = \lim_{k\to+\infty} \sum_{n=0}^{k} \varphi(J_n)(x) = \sum_{n=0}^{+\infty} \varphi(J_n)(x)$.

As $J_k \cap J_l = \emptyset$ $\forall x \in [0;a]$ $\forall k, l \in \mathbb{N}, \sum_{n=0}^{+\infty} \varphi(J_n)(x) \leq -\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dy$. or $-\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dy$ exist and is finite then $S(x)$ exists and is finite. then the sequence of partial sums $(S_k)_{k\in\mathbb{N}}$ admits a finite limit so it converges to S on $[0;a]$. Thus the series of functions defined on $[0;a]$, $\sum_k \varphi(J_k)(x)$, simply converge to S. $\forall x \in [0;a]$ the series with general term $\varphi(J_k)$, $\sum_k \varphi(J_k)(x)$, is a convergent series with positive terms so its general term $\varphi(J_k)$ tends to zero when k tends to $+\infty$.But there exists $z_k \in J_k$ such that $\varphi_k(x) = -\int_{y_k}^{y_{k+1}} \overrightarrow{\nabla} \cdot \overrightarrow{H}(x, y) dy = -(y_{k+1} - y_k) \overrightarrow{\nabla} \cdot \overrightarrow{H}(x, z_k)$. The sequence $(z_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive real numbers tending towards $+\infty$. Let $u_k = y_{k+1} - y_k$. If the sequence $(u_k)_{k\in\mathbb{N}}$ converges to zero when k tends to $+\infty$ then $\forall\varepsilon > 0$, $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}$, $k > k_0 \implies |y_{k+1} - y_k| < \varepsilon$. Which means that the sequence $(y_k)_{k\in\mathbb{N}}$ is from cauchy. Which is contradictory to the fact that it is divergent. Consequently, $\forall x \in [-a, a], \varphi_k(x)$ tends to zero when k tends to $+\infty$ if and only if $\overrightarrow{\nabla} \cdot \overrightarrow{H}(x, z_k)$ tends to zero as k tends to $+\infty$. But

$$\begin{aligned}
&\overrightarrow{\nabla} \cdot \overrightarrow{H}(x, z_k) \\
= \quad & \sqrt{2}cs \ln\left(\frac{N_2(x, z_k) N_4(x, z_k)}{N_1(x, z_k) N_8(x, z_k)}\right)\left(1 - \frac{N_2(x, z_k) N_4(x, z_k)}{N_1(x, z_k) N_8(x, z_k)}\right) N_1(x, z_k) N_8(x, z_k) \\
+ \quad & \sqrt{2}cs \ln\left(\frac{N_2(x, z_k) N_7(x, z_k)}{N_1(x, z_k) N_9(x, z_k)}\right)\left(1 - \frac{N_2(x, z_k) N_7(x, z_k)}{N_1(x, z_k) N_9(x, z_k)}\right) N_1(x, z_k) N_9(x, z_k)
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{2}cs \ln\left(\frac{N_3(x,z_k)\,N_4(x,z_k)}{N_1(x,z_k)\,N_{12}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_4(x,z_k)}{N_1(x,z_k)\,N_{12}(x,z_k)}\right)N_1(x,z_k)\,N_{12}(x,z_k)\\
&+ \sqrt{2}cs \ln\left(\frac{N_3(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{13}(x,z_k)}\right)N_1(x,z_k)\,N_{13}(x,z_k)\\
&+ \sqrt{2}cs \ln\left(\frac{N_5(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{10}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{10}(x,z_k)}\right)N_1(x,z_k)\,N_{10}(x,z_k)\\
&+ \sqrt{2}cs \ln\left(\frac{N_4(x,z_k)\,N_5(x,z_k)}{N_1(x,z_k)\,N_{11}(x,z_k)}\right)\left(1-\frac{N_4(x,z_k)\,N_5(x,z_k)}{N_1(x,z_k)\,N_{11}(x,z_k)}\right)N_1(x,z_k)\,N_{11}(x,z_k)\\
&+ \sqrt{2}cs \ln\left(\frac{N_4(x,z_k)\,N_6(x,z_k)}{N_1(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_4(x,z_k)\,N_6(x,z_k)}{N_1(x,z_k)\,N_{15}(x,z_k)}\right)N_1(x,z_k)\,N_{15}(x,z_k)\\
&+ \sqrt{2}cs \ln\left(\frac{N_6(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{14}(x,z_k)}\right)\left(1-\frac{N_6(x,z_k)\,N_7(x,z_k)}{N_1(x,z_k)\,N_{14}(x,z_k)}\right)N_1(x,z_k)\,N_{14}(x,z_k)\\
&+ 2cs \ln\left(\frac{N_2(x,z_k)\,N_5(x,z_k)}{N_3(x,z_k)\,N_6(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_5(x,z_k)}{N_3(x,z_k)\,N_6(x,z_k)}\right)N_3(x,z_k)\,N_6(x,z_k)\\
&+ 2cs \ln\left(\frac{N_2(x,z_k)\,N_5(x,z_k)}{N_4(x,z_k)\,N_7(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_5(x,z_k)}{N_4(x,z_k)\,N_7(x,z_k)}\right)N_4(x,z_k)\,N_7(x,z_k)\\
&+ 2cs \ln\left(\frac{N_3(x,z_k)\,N_6(x,z_k)}{N_4(x,z_k)\,N_7(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_6(x,z_k)}{N_4(x,z_k)\,N_7(x,z_k)}\right)N_4(x,z_k)\,N_7(x,z_k)\\
&+ 2cs \ln\left(\frac{N_8(x,z_k)\,N_{11}(x,z_k)}{N_{12}(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{11}(x,z_k)}{N_{12}(x,z_k)\,N_{15}(x,z_k)}\right)N_{12}(x,z_k)\,N_{15}(x,z_k)\\
&+ 2cs \ln\left(\frac{N_9(x,z_k)\,N_{10}(x,z_k)}{N_{13}(x,z_k)\,N_{14}(x,z_k)}\right)\left(1-\frac{N_9(x,z_k)\,N_{10}(x,z_k)}{N_{13}(x,z_k)\,N_{14}(x,z_k)}\right)N_{13}(x,z_k)\,N_{14}(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_2(x,z_k)\,N_{12}(x,z_k)}{N_3(x,z_k)\,N_8(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{12}(x,z_k)}{N_3(x,z_k)\,N_8(x,z_k)}\right)N_3(x,z_k)\,N_8(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_2(x,z_k)\,N_{13}(x,z_k)}{N_3(x,z_k)\,N_9(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{13}(x,z_k)}{N_3(x,z_k)\,N_9(x,z_k)}\right)N_3(x,z_k)\,N_9(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_2(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_8(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_8(x,z_k)}\right)N_6(x,z_k)\,N_8(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_2(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_9(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_9(x,z_k)}\right)N_6(x,z_k)\,N_9(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N(x,z_k)\,N_{12}(x,z_k)}{N_3(x,z_k)\,N_{11}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{12}(x,z_k)}{N_3(x,z_k)\,N_{11}(x,z_k)}\right)N_3(x,z_k)\,N_{11}(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_5(x,z_k)\,N_{13}(x,z_k)}{N_3(x,z_k)\,N_{10}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{13}(x,z_k)}{N_3(x,z_k)\,N_{10}(x,z_k)}\right)N_3(x,z_k)\,N_{10}(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_5(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_{10}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_{10}(x,z_k)}\right)N_6(x,z_k)\,N_{10}(x,z_k)\\
&+ \sqrt{3}cs \ln\left(\frac{N_5(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_{11}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_{11}(x,z_k)}\right)N_6(x,z_k)\,N_{11}(x,z_k)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_3(x,z_k)\,N_{14}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_3(x,z_k)\,N_{14}(x,z_k)}\right)N_3(x,z_k)\,N_{14}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_3(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_3(x,z_k)\,N_{15}(x,z_k)}\right)N_3(x,z_k)\,N_{15}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_5(x,z_k)\,N_{8}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_5(x,z_k)\,N_{8}(x,z_k)}\right)N_5(x,z_k)\,N_{8}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_5(x,z_k)\,N_{9}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_5(x,z_k)\,N_{9}(x,z_k)}\right)N_5(x,z_k)\,N_{9}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{11}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)N_6(x,z_k)\,N_{12}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_{10}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)N_6(x,z_k)\,N_{13}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_{14}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)N_6(x,z_k)\,N_{13}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(x,z_k)\,N_{14}(x,z_k)}{N_5(x,z_k)\,N_{9}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_{14}(x,z_k)}{N_5(x,z_k)\,N_{9}(x,z_k)}\right)N_5(x,z_k)\,N_{9}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(x,z_k)\,N_{15}(x,z_k)}{N_5(x,z_k)\,N_{8}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_{15}(x,z_k)}{N_5(x,z_k)\,N_{8}(x,z_k)}\right)N_5(x,z_k)\,N_{8}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)\left(1-\frac{N_3(x,z_k)\,N_{15}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)N_6(x,z_k)\,N_{12}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_5(x,z_k)\,N_{8}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{8}(x,z_k)}{N_6(x,z_k)\,N_{12}(x,z_k)}\right)N_6(x,z_k)\,N_{12}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_5(x,z_k)\,N_{9}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_5(x,z_k)\,N_{9}(x,z_k)}{N_6(x,z_k)\,N_{13}(x,z_k)}\right)N_6(x,z_k)\,N_{13}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_7(x,z_k)\,N_{8}(x,z_k)}{N_4(x,z_k)\,N_{9}(x,z_k)}\right)\left(1-\frac{N_7(x,z_k)\,N_{8}(x,z_k)}{N_4(x,z_k)\,N_{9}(x,z_k)}\right)N_4(x,z_k)\,N_{9}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_4(x,z_k)\,N_{10}(x,z_k)}{N_7(x,z_k)\,N_{11}(x,z_k)}\right)\left(1-\frac{N_4(x,z_k)\,N_{10}(x,z_k)}{N_7(x,z_k)\,N_{11}(x,z_k)}\right)N_7(x,z_k)\,N_{11}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_7(x,z_k)\,N_{12}(x,z_k)}{N_4(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_7(x,z_k)\,N_{12}(x,z_k)}{N_4(x,z_k)\,N_{13}(x,z_k)}\right)N_4(x,z_k)\,N_{13}(x,z_k)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_4(x,z_k)\,N_{14}(x,z_k)}{N_7(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_4(x,z_k)\,N_{14}(x,z_k)}{N_7(x,z_k)\,N_{15}(x,z_k)}\right)N_7(x,z_k)\,N_{15}(x,z_k)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_8(x,z_k)\,N_{13}(x,z_k)}{N_9(x,z_k)\,N_{12}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{13}(x,z_k)}{N_9(x,z_k)\,N_{12}(x,z_k)}\right)N_9(x,z_k)\,N_{12}(x,z_k)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_8(x,z_k)\,N_{14}(x,z_k)}{N_9(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{14}(x,z_k)}{N_9(x,z_k)\,N_{15}(x,z_k)}\right)N_9(x,z_k)\,N_{15}(x,z_k)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_{10}(x,z_k)\,N_{12}(x,z_k)}{N_{11}(x,z_k)\,N_{13}(x,z_k)}\right)\left(1-\frac{N_{10}(x,z_k)\,N_{12}(x,z_k)}{N_{11}(x,z_k)\,N_{13}(x,z_k)}\right)N_{11}(x,z_k)\,N_{13}(x,z_k)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_{11}(x,z_k)\,N_{14}(x,z_k)}{N_{15}(x,z_k)\,N_{10}(x,z_k)}\right)\left(1-\frac{N_{11}(x,z_k)\,N_{14}(x,z_k)}{N_{15}(x,z_k)\,N_{10}(x,z_k)}\right)N_{15}(x,z_k)\,N_{10}(x,z_k)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_9(x,z_k)\,N_{11}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_9(x,z_k)\,N_{11}(x,z_k)}\right)N_9(x,z_k)\,N_{11}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_{12}(x,z_k)\,N_{14}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_{12}(x,z)\,N_{14}(x,z_k)}\right)N_{12}(x,z_k)\,N_{14}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_8(x,z_k)\,N_{10}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)N_{13}(x,z_k)\,N_{15}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_9(x,z_k)\,N_{11}(x,z_k)}{N_{12}(x,z_k)\,N_{14}(x,z_k)}\right)\left(1-\frac{N_9(x,z_k)\,N_{11}(x,z_k)}{N_{12}(x,z_k)\,N_{14}(x,z_k)}\right)N_{12}(x,z_k)\,N_{14}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_9(x,z_k)\,N_{11}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_9(x,z_k)\,N_{11}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)N_{13}(x,z_k)\,N_{15}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_{12}(x,z_k)\,N_{14}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)\left(1-\frac{N_{12}(x,z_k)\,N_{14}(x,z_k)}{N_{13}(x,z_k)\,N_{15}(x,z_k)}\right)N_{13}(x,z_k)\,N_{15}(x,z_k).
\end{aligned}
$$

So, if $N_\alpha^M(x)=\lim\limits_{k\longrightarrow+\infty}N_\alpha(x,z_k)\,,\alpha\in\{1,2,3,\ldots,15\}$ then by setting $a=\sqrt{2}cs,b=2cs,d=\sqrt{3}cs,e=\sqrt{5}cs,f=\sqrt{6}cs,g=\sqrt{2}cs:$

$$
\begin{aligned}
&+ \quad a\ln\left(\frac{N_2^M(x)\,N_4^M(x)}{N_1^M(x)\,N_8^M(x)}\right)\left(1-\frac{N_2^M(x)\,N_4^M(x)}{N_1^M(x)\,N_8^M(x)}\right)N_1^M(x)\,N_8^M(x)\\
&+ \quad a\ln\left(\frac{N_2^M(x)\,N_7^M(x)}{N_1^M(x)\,N_9^M(x)}\right)\left(1-\frac{N_2^M(x)\,N_7^M(x)}{N_1^M(x)\,N_9^M(x)}\right)N_1^M(x)\,N_9^M(x)\\
&+ \quad a\ln\left(\frac{N_3^M(x)\,N_4^M(x)}{N_1^M(x)\,N_{12}^M(x)}\right)\left(1-\frac{N_3^M(x)\,N_4^M(x)}{N_1^M(x)\,N_{12}^M(x)}\right)N_1^M(x)\,N_{12}^M(x)\\
&+ \quad a\ln\left(\frac{N_3^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{13}^M(x)}\right)\left(1-\frac{N_3^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{13}^M(x)}\right)N_1^M(x)\,N_{13}^M(x)\\
&+ \quad a\ln\left(\frac{N_5^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{10}^M(x)}\right)\left(1-\frac{N_5^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{10}^M(x)}\right)N_1^M(x)\,N_{10}^M(x)\\
&+ \quad a\ln\left(\frac{N_4^M(x)\,N_5^M(x)}{N_1^M(x)\,N_{11}^M(x)}\right)\left(1-\frac{N_4^M(x)\,N_5^M(x)}{N_1^M(x)\,N_{11}^M(x)}\right)N_1^M(x)\,N_{11}^M(x)\\
&+ \quad a\ln\left(\frac{N_4^M(x)\,N_6^M(x)}{N_1^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_4^M(x)\,N_6^M(x)}{N_1^M(x)\,N_{15}^M(x)}\right)N_1^M(x)\,N_{15}^M(x)\\
&+ \quad a\ln\left(\frac{N_6^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{14}^M(x)}\right)\left(1-\frac{N_6^M(x)\,N_7^M(x)}{N_1^M(x)\,N_{14}^M(x)}\right)N_1^M(x)\,N_{14}^M(x)\\
&+ \quad b\ln\left(\frac{N_2^M(x)\,N_5^M(x)}{N_3^M(x)\,N_6^M(x)}\right)\left(1-\frac{N_2^M(x)\,N_5^M(x)}{N_3^M(x)\,N_6^M(x)}\right)N_3^M(x)\,N_6^M(x)\\
&+ \quad b\ln\left(\frac{N_2^M(x)\,N_5^M(x)}{N_4^M(x)\,N_7^M(x)}\right)\left(1-\frac{N_2^M(x)\,N_5^M(x)}{N_4^M(x)\,N_7^M(x)}\right)N_4^M(x)\,N_7^M(x)\\
&+ \quad b\ln\left(\frac{N_3^M(x)\,N_6^M(x)}{N_4^M(x)\,N_7^M(x)}\right)\left(1-\frac{N_3^M(x)\,N_6^M(x)}{N_4^M(x)\,N_7^M(x)}\right)N_4^M(x)\,N_7^M(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad b \ln\left(\frac{N_8^M(x)\, N_{11}^M(x)}{N_{12}^M(x)\, N_{15}^M(x)}\right)\left(1-\frac{N_8^M(x)\, N_{11}^M(x)}{N_{12}^M(x)\, N_{15}^M(x)}\right) N_{12}^M(x)\, N_{15}^M(x)\\
&+ \quad b \ln\left(\frac{N_9^M(x)\, N_{10}^M(x)}{N_{13}^M(x)\, N_{14}^M(x)}\right)\left(1-\frac{N_9^M(x)\, N_{10}^M(x)}{N_{13}^M(x)\, N_{14}^M(x)}\right) N_{13}^M(x)\, N_{14}^M(x)\\
&+ \quad d \ln\left(\frac{N_2^M(x)\, N_{12}^M(x)}{N_3^M(x)\, N_8^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{12}^M(x)}{N_3^M(x)\, N_8^M(x)}\right) N_3^M(x)\, N_8^M(x)\\
&+ \quad d \ln\left(\frac{N_2^M(x)\, N_{13}^M(x)}{N_3^M(x)\, N_9^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{13}^M(x)}{N_3^M(x)\, N_9^M(x)}\right) N_3^M(x)\, N_9^M(x)\\
&+ \quad d \ln\left(\frac{N_2^M(x)\, N_{15}^M(x)}{N_6^M(x)\, N_8^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{15}^M(x)}{N_6^M(x)\, N_8^M(x)}\right) N_6^M(x)\, N_8^M(x)\\
&+ \quad d \ln\left(\frac{N_2^M(x)\, N_{14}^M(x)}{N_6^M(x)\, N_9^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{14}^M(x)}{N_6^M(x)\, N_9^M(x)}\right) N_6^M(x)\, N_9^M(x)\\
&+ \quad d \ln\left(\frac{N_5^M(x)\, N_{12}^M(x)}{N_3^M(x)\, N_{11}^M(x)}\right)\left(1-\frac{N_5^M(x)\, N_{12}^M(x)}{N_3^M(x)\, N_{11}^M(x)}\right) N_3^M(x)\, N_{11}^M(x)\\
&+ \quad d \ln\left(\frac{N_5^M(x)\, N_{13}^M(x)}{N_3^M(x)\, N_{10}^M(x)}\right)\left(1-\frac{N_5^M(x)\, N_{13}^M(x)}{N_3^M(x)\, N_{10}^M(x)}\right) N_3^M(x)\, N_{10}^M(x)\\
&+ \quad d \ln\left(\frac{N_5^M(x)\, N_{14}^M(x)}{N_6^M(x)\, N_{10}^M(x)}\right)\left(1-\frac{N_5^M(x)\, N_{14}^M(x)}{N_6^M(x)\, N_{10}^M(x)}\right) N_6^M(x)\, N_{10}^M(x)\\
&+ \quad d \ln\left(\frac{N_5^M(x)\, N_{15}^M(x)}{N_6^M(x)\, N_{11}^M(x)}\right)\left(1-\frac{N_5^M(x)\, N_{15}^M(x)}{N_6^M(x)\, N_{11}^M(x)}\right) N_6^M(x)\, N_{11}^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{10}^M(x)}{N_3^M(x)\, N_{14}^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{10}^M(x)}{N_3^M(x)\, N_{14}^M(x)}\right) N_3^M(x)\, N_{14}^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{11}^M(x)}{N_3^M(x)\, N_{15}^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{11}^M(x)}{N_3^M(x)\, N_{15}^M(x)}\right) N_3^M(x)\, N_{15}^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{11}^M(x)}{N_5^M(x)\, N_8^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{11}^M(x)}{N_5^M(x)\, N_8^M(x)}\right) N_5^M(x)\, N_8^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{10}^M(x)}{N_5^M(x)\, N_9^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{10}^M(x)}{N_5^M(x)\, N_9^M(x)}\right) N_5^M(x)\, N_9^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{11}^M(x)}{N_6^M(x)\, N_{12}^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{11}^M(x)}{N_6^M(x)\, N_{12}^M(x)}\right) N_6^M(x)\, N_{12}^M(x)\\
&+ \quad e \ln\left(\frac{N_2^M(x)\, N_{10}^M(x)}{N_6^M(x)\, N_{13}^M(x)}\right)\left(1-\frac{N_2^M(x)\, N_{10}^M(x)}{N_6^M(x)\, N_{13}^M(x)}\right) N_6^M(x)\, N_{13}^M(x)\\
&+ \quad e \ln\left(\frac{N_3^M(x)\, N_{14}^M(x)}{N_6^M(x)\, N_{13}^M(x)}\right)\left(1-\frac{N_3^M(x)\, N_{114}^M(x)}{N_6^M(x)\, N_{13}^M(x)}\right) N_6^M(x)\, N_{13}^M(x)\\
&+ \quad e \ln\left(\frac{N_3^M(x)\, N_{14}^M(x)}{N_5^M(x)\, N_9^M(x)}\right)\left(1-\frac{N_3^M(x)\, N_{14}^M(x)}{N_5^M(x)\, N_9^M(x)}\right) N_5^M(x)\, N_9^M(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad e\ln\left(\frac{N_3^M(x)\,N_{15}^M(x)}{N_5^M(x)\,N_8^M(x)}\right)\left(1-\frac{N_3^M(x)\,N_{15}^M(x)}{N_5^M(x)\,N_8^M(x)}\right)N_5^M(x)\,N_8^M(x)\\
&+ \quad e\ln\left(\frac{N_3^M(x)\,N_{15}^M(x)}{N_6^M(x)\,N_{12}^M(x)}\right)\left(1-\frac{N_3^M(x)\,N_{15}^M(x)}{N_6^M(x)\,N_{12}^M(x)}\right)N_6^M(x)\,N_{12}^M(x)\\
&+ \quad e\ln\left(\frac{N_5^M(x)\,N_8^M(x)}{N_6^M(x)\,N_{12}^M(x)}\right)\left(1-\frac{N_5^M(x)\,N_8^M(x)}{N_6^M(x)\,N_{12}^M(x)}\right)N_6^M(x)\,N_{12}^M(x)\\
&+ \quad e\ln\left(\frac{N_5^M(x)\,N_9^M(x)}{N_6^M(x)\,N_{13}^M(x)}\right)\left(1-\frac{N_5^M(x)\,N_9^M(x)}{N_6^M(x)\,N_{13}^M(x)}\right)N_6^M(x)\,N_{13}^M(x)\\
&+ \quad e\ln\left(\frac{N_7^M(x)\,N_8^M(x)}{N_4^M(x)\,N_9^M(x)}\right)\left(1-\frac{N_7^M(x)\,N_8^M(x)}{N_4^M(x)\,N_9^M(x)}\right)N_4^M(x)\,N_9^M(x)\\
&+ \quad e\ln\left(\frac{N_4^M(x)\,N_{10}^M(x)}{N_7^M(x)\,N_{11}^M(x)}\right)\left(1-\frac{N_4^M(x)\,N_{10}^M(x)}{N_7^M(x)\,N_{11}^M(x)}\right)N_7^M(x)\,N_{11}^M(x)\\
&+ \quad e\ln\left(\frac{N_7^M(x)\,N_{12}^M(x)}{N_4^M(x)\,N_{13}^M(x)}\right)\left(1-\frac{N_7^M(x)\,N_{12}^M(x)}{N_4^M(x)\,N_{13}^M(x)}\right)N_4^M(x)\,N_{13}^M(x)\\
&+ \quad e\ln\left(\frac{N_4^M(x)\,N_{14}^M(x)}{N_7^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_4^M(x)\,N_{14}^M(x)}{N_7^M(x)\,N_{15}^M(x)}\right)N_7^M(x)\,N_{15}^M(x)\\
&+ \quad f\ln\left(\frac{N_8^M(x)\,N_{13}^M(x)}{N_9^M(x)\,N_{12}^M(x)}\right)\left(1-\frac{N_8^M(x)\,N_{13}^M(x)}{N_9^M(x)\,N_{12}^M(x)}\right)N_9^M(x)\,N_{12}^M(x)\\
&+ \quad f\ln\left(\frac{N_8^M(x)\,N_{14}^M(x)}{N_9^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_8^M(x)\,N_{14}^M(x)}{N_9^M(x)\,N_{15}^M(x)}\right)N_9^M(x)\,N_{15}^M(x)\\
&+ \quad f\ln\left(\frac{N_{10}^M(x)\,N_{12}^M(x)}{N_{11}^M(x)\,N_{13}^M(x)}\right)\left(1-\frac{N_{10}^M(x)\,N_{12}^M(x)}{N_{11}^M(x)\,N_{13}^M(x)}\right)N_{11}^M(x)\,N_{13}^M(x)\\
&+ \quad f\ln\left(\frac{N_{11}^M(x)\,N_{14}^M(x)}{N_{15}^M(x)\,N_{10}^M(x)}\right)\left(1-\frac{N_{11}^M(x)\,N_{14}^M(x)}{N_{15}^M(x)\,N_{10}^M(x)}\right)N_{15}^M(x)\,N_{10}^M(x)\\
&+ \quad g\ln\left(\frac{N_8^M(x)\,N_{10}^M(x)}{N_9^M(x)\,N_{11}^M(x)}\right)\left(1-\frac{N_8^M(x)\,N_{10}^M(x)}{N_9^M(x)\,N_{11}^M(x)}\right)N_9^M(x)\,N_{11}^M(x)\\
&+ \quad g\ln\left(\frac{N_8^M(x)\,N_{10}^M(x)}{N_{12}^M(x)\,N_{14}^M(x)}\right)\left(1-\frac{N_8^M(x)\,N_{10}^M(x)}{N_{12}^M(x)\,N_{14}^M(x)}\right)N_{12}^M(x)\,N_{14}^M(x)\\
&+ \quad g\ln\left(\frac{N_8^M(x)\,N_{10}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_8^M(x)\,N_{10}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)N_{13}^M(x)\,N_{15}^M(x)\\
&+ \quad g\ln\left(\frac{N_9^M(x)\,N_{11}^M(x)}{N_{12}^M(x)\,N_{14}^M(x)}\right)\left(1-\frac{N_9^M(x)\,N_{11}^M(x)}{N_{12}^M(x)\,N_{14}^M(x)}\right)N_{12}^M(x)\,N_{14}^M(x)\\
&+ \quad g\ln\left(\frac{N_9^M(x)\,N_{11}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_9^M(x)\,N_{11}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)N_{13}^M(x)\,N_{15}^M(x)\\
&+ \quad g\ln\left(\frac{N_{12}^M(x)\,N_{14}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)\left(1-\frac{N_{12}^M(x)\,N_{14}^M(x)}{N_{13}^M(x)\,N_{15}^M(x)}\right)N_{13}^M(x)\,N_{15}^M(x)=0
\end{aligned}
$$

Which is only possible for positive $N_\alpha^M(x)\,,\alpha=1,2,3,\ldots,15$ if

$N_3^M(x)\,N_4^M(x) = N_1^M(x)\,N_{12}^M(x)$, $N_3^M(x)\,N_7^M(x) = N_1^M(x)\,N_{13}^M(x)$,
$N_5^M(x)\,N_7^M(x) = N_1^M(x)\,N_{10}^M(x)$, $N_4^M(x)\,N_5^M(x) = N_1^M(x)\,N_{11}^M(x)$,
$N_4^M(x)\,N_6^M(x) = N_1^M(x)\,N_{15}^M(x)$, $N_6^M(x)\,N_7^M(x) = N_1^M(x)\,N_{14}^M(x)$,
$N_2^M(x)\,N_5^M(x) = N_3^M(x)\,N_6^M(x)$, $N_2^M(x)\,N_5^M(x) = N_4^M(x)\,N_7^M(x)$,
$N_3^M(x)\,N_6^M(x) = N_4^M(x)\,N_7^M(x)$, $N_8^M(x)\,N_{11}^M(x) = N_{12}^M(x)\,N_{15}^M(x)$,
$N_9^M(x)\,N_{10}^M(x) = N_{13}^M(x)\,N_{14}^M(x)$, $N_2^M(x)\,N_4^M(x) = N_1^M(x)\,N_8^M(x)$,
$N_2^M(x)\,N_7^M(x) = N_1^M(x)\,N_9^M(x)$, $N_2^M(x)\,N_{12}^M(x) = N_3^M(x)\,N_8^M(x)$,
$N_2^M(x)\,N_{13}^M(x) = N_3^M(x)\,N_9^M(x)$, $N_2^M(x)\,N_{15}^M(x) = N_6^M(x)\,N_8^M(x)$,
$N_2^M(x)\,N_{14}^M(x) = N_6^M(x)\,N_9^M(x)$, $N_5^M(x)\,N_{12}^M(x) = N_3^M(x)\,N_{11}^M(x)$,
$N_5^M(x)\,N_{13}^M(x) = N_3^M(x)\,N_{10}^M(x)$, $N_5^M(x)\,N_{14}^M(x) = N_6^M(x)\,N_{10}^M(x)$,
$N_5^M(x)\,N_{15}^M(x) = N_6^M(x)\,N_{11}^M(x)$, $N_8^M(x)\,N_{13}^M(x) = N_9^M(x)\,N_{12}^M(x)$,
$N_8^M(x)\,N_{14}^M(x) = N_9^M(x)\,N_{15}^M(x)$, $N_{10}^M(x)\,N_{12}^M(x) = N_{11}^M(x)\,N_{13}^M(x)$,
$N_{11}^M(x)\,N_{14}^M(x) = N_{10}^M(x)\,N_{15}^M(x)$, $N_8^M(x)\,N_{10}^M(x) = N_9^M(x)\,N_{11}^M(x)$,
$N_8^M(x)\,N_{10}^M(x) = N_{12}^M(x)\,N_{14}^M(x)$, $N_8^M(x)\,N_{10}^M(x) = N_{13}^M(x)\,N_{15}^M(x)$,
$N_9^M(x)\,N_{11}^M(x) = N_{12}^M(x)\,N_{14}^M(x)$, $N_9^M(x)\,N_{11}^M(x) = N_{13}^M(x)\,N_{15}^M(x)$,
$N_{12}^M(x)\,N_{14}^M(x) = N_{13}^M(x)\,N_{15}^M(x)$, $N_2^M(x)\,N_{10}^M(x) = N_3^M(x)\,N_{14}^M(x)$,
$N_2^M(x)\,N_{11}^M(x) = N_3^M(x)\,N_{15}^M(x)$, $N_2^M(x)\,N_{11}^M(x) = N_5^M(x)\,N_8^M(x)$,
$N_2^M(x)\,N_{10}^M(x) = N_5^M(x)\,N_9^M(x)$, $N_2^M(x)\,N_{11}^M(x) = N_6^M(x)\,N_{12}^M(x)$,
$N_2^M(x)\,N_{10}^M(x) = N_6^M(x)\,N_{13}^M(x)$, $N_3^M(x)\,N_{14}^M(x) = N_6^M(x)\,N_{13}^M(x)$,
$N_3^M(x)\,N_{14}^M(x) = N_5^M(x)\,N_9^M(x)$, $N_3^M(x)\,N_{15}^M(x) = N_5^M(x)\,N_8^M(x)$,
$N_3^M(x)\,N_{15}^M(x) = N_6^M(x)\,N_{12}^M(x)$, $N_5^M(x)\,N_8^M(x) = N_6^M(x)\,N_{12}^M(x)$,
$N_5^M(x)\,N_9^M(x) = N_6^M(x)\,N_{13}^M(x)$, $N_7^M(x)\,N_8^M(x) = N_4^M(x)\,N_9^M(x)$,
$N_4^M(x)\,N_{10}^M(x) = N_7^M(x)\,N_{11}^M(x)$, $N_7^M(x)\,N_{12}^M(x) = N_4^M(x)\,N_{13}^M(x)$,
$N_4^M(x)\,N_{14}^M(x) = N_7^M(x)\,N_{15}^M(x)$,
i.e N^M is a Maxwellian equilibrium state. □

Proposition 3.3. *For any decreasing sequence $(y_k)_{k\in\mathbb{N}}$ of negative real numbers tending towards $-\infty$ the sequence $(N(x,y_k))_{k\in\mathbb{N}}$ tends to a Maxwellian state when k tends to $+\infty$ $\forall x\in[0,a]$.*

Proof. This proof is done in the same way as that of proposition (3.2) □

We have shown that the boundary value problem (2.1)-(3.1) has uniformly continuous and bounded solutions independently of a on $[0;a]\times]-\infty;+\infty[$ and that these solutions tend towards a Maxwellian state at infinity. Thus let $(a_l)_{l\in\mathbb{N}}$ be an increasing sequence of positive real numbers tending towards $+\infty$, the solutions

N^l of (2.1)-(3.1) on $[0,a_l]\times]-\infty;+\infty[$ are uniformly continuous and bounded independently of a_l and their set is also equicontinuous for all $l\in\mathbb{N}$. We extend N^l to $[0,+\infty[\times]-\infty,+\infty[$ by the function $\widetilde{G}^l$ defined by:

$$\widetilde{G}^l(x,y)=\begin{cases}N^l(x,y) & ,(x,y)\in[0,a_l]\times\,]-\infty,+\infty[\\ N^l(a_l,y) & ,(x,y)\in[0,+\infty[\,\times\mathbb{R}, x>a_l\end{cases}.$$

Here, $\left(\widetilde{G}^l\right)_{l\in\mathbb{N}}$ is a sequence of uniformly continuous and bounded functions satisfying (2.1)-(2.2).

Proposition 3.4. *There exist a subsequence of* $\left(\widetilde{G}^l\right)_{l\in\mathbb{N}}$ *which simply converges on* $[0;+\infty[\times]-\infty;+\infty[$ *and uniformly on any strip from* $[0;+\infty[\times]-\infty;+\infty[$ *towards a uniformly continuous and bounded function* G

Proof. The function $\widetilde{G}^l$ are bounded on $[0,+\infty[\times]-\infty;+\infty[$ therefore $\forall k\in\Lambda$, $\forall(x,y)\in[0;+\infty[\times]-\infty;+\infty[$, the sequence $\left(\widetilde{G}^l_k(x,y)\right)_{l\in\mathbb{N}}$ is a bounded sequence of positive real numbers. We can therefore extract a convergent subsequence $\left(\widetilde{G}^{l_k}_k\right)_{l_k\in\mathbb{N}}$. Let $\widetilde{G}_k(x,y)$ be the limit of this subsequence. $\forall k\in\Lambda$, we define on $[0;+\infty[\times]-\infty;+\infty[$ the function $G=(G_k)_{k\in\Lambda}$ by $G_k(x,y)=\widetilde{G}_k(x,y)$. The sequence of functions $\left(\widetilde{G}^l\right)_{l\in\mathbb{N}}$ simply converges to G on $[0;+\infty[\times]-\infty;+\infty[$. Moreover G is uniformly continuous because $\forall(x,y),(x',y')\in[0;a]\times\,]-\infty,+\infty[$,

$$|G_k(x,y)-G_k(x',y')|\le\left|\widetilde{G}^l_k(x,y)-G_k(x,y)\right|$$
$$+\left|\widetilde{G}^l_k(x,y)-\widetilde{G}^l_k(x',y')\right|+\left|G_k(x',y')-\widetilde{G}^l_k(x',y')\right|.$$

But since $\left(\widetilde{G}^l\right)_{l\in\mathbb{N}}$ simply converges to G on $[0;+\infty[\,\times\,]-\infty,+\infty[$ and $\widetilde{G}^l_k$ is uniformly continuous $\forall l\in\mathbb{N}$ and $\forall k\in\Lambda$ we have: $\forall\varepsilon>0$, $\exists l_0\in\mathbb{N}$, $\exists\alpha>0$, $\forall(x,y),(x',y')\in[0;+\infty[\,\times\,]-\infty,+\infty[$, $\forall l,l>l_0$ $|x-x'|<\alpha$, $|y-y'|<\alpha\implies\left|\widetilde{G}^l_k(x,y)-G_k(x,y)\right|<\frac{\varepsilon}{3}$, $\left|\widetilde{G}^l_k(x,y)-\widetilde{G}^l_k(x',y')\right|<\frac{\varepsilon}{3}$, $\left|G_k(x',y')-\widetilde{G}^l_k(x',y')\right|<\frac{\varepsilon}{3}$.

We finally show that $\left(\widetilde{G}^l_k\right)_{l\in\mathbb{N}}$ converges uniformly towards G_k on any strip of $[0;+\infty[\,\times\mathbb{R}$ by noting that $\forall(x,y)\in[0;+\infty[\times]-\infty,+\infty[$, $\exists(x',y')\in[0;+\infty[\times]-$

$\infty, +\infty[$ such that

$$\left|\widetilde{G}_k^l\left(x,y\right) - G_k\left(x,y\right)\right| \le \left|\widetilde{G}_k^l(x,y) - \widetilde{G}_k^l(x',y')\right| + \left|\widetilde{G}_k^l\left(x',y'\right) - G_k\left(x',y'\right)\right| + \left|G_k(x',y') - G_k(x,y)\right|$$

$\square$

Proposition 3.5. *For any increasing sequence $(x_k)_{k\in\mathbb{N}}$ of positive real numbers tending towards $+\infty$ the sequence $(N(x_k,y))_{k\in\mathbb{N}}$ tends to a Maxwellian state when k tends to $+\infty$ $\forall y \in]-\infty;+\infty[$.*

Proof. We set $\overrightarrow{u}_k = u_k^\alpha \overrightarrow{e}_\alpha, h_\alpha(x,y) = \sum_{k=1}^{15} N_k(x,y)\ln N_k(x,y)\, u_k^\alpha$ and $\overrightarrow{H} = \sum_{\alpha=1}^{2} h_\alpha \overrightarrow{e}_\alpha$, h_α is a continuous function on $[0;+\infty[\times]-\infty;+\infty[$, $\forall\alpha$. We have:

$$\begin{aligned}
& \overrightarrow{\nabla}\cdot\overrightarrow{H} = \frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \\
= \;& c\left(1+\ln N_2\right)\frac{\partial N_2}{\partial x} - c\left(1+\ln N_5\right)\frac{\partial N_5}{\partial x} + c\left(1+\ln N_8\right)\frac{\partial N_8}{\partial x} \\
+ \;& c\left(1+\ln N_9\right)\tfrac{\partial N_9}{\partial x} - c\left(1+\ln N_{10}\right)\frac{\partial N_{10}}{\partial x} - c\left(1+\ln N_{11}\right)\frac{\partial N_{11}}{\partial x} \\
+ \;& c\left(1+\ln N_3\right)\frac{\partial N_3}{\partial y} - c\left(1+\ln N_6\right)\frac{\partial N_6}{\partial y} + c\left(1+\ln N_{12}\right)\frac{\partial N_{12}}{\partial y} \\
+ \;& c\left(1+\ln N_{13}\right)\frac{\partial N_{13}}{\partial y} - c\left(1+\ln N_{14}\right)\frac{\partial N_{14}}{\partial y} - c\left(1+\ln N_{15}\right)\frac{\partial N_{15}}{\partial y}.
\end{aligned}$$

By multiplying the first equation of (2.1) par $1+\ln N_1$, the second by $1+\ln N_2$,..., the k^{eme} by $1+\ln N_k$ and by making a member-by-member sum, we obtain (3.10). The function $z \longmapsto (1-z)\ln z$ is negative or zero for $z>0$ (zero for $z=1$). As $\sqrt{2}cs, 2cs, \sqrt{3}cs, \sqrt{5}cs, \sqrt{6}cs, 2\sqrt{2}cs$ are positive as well as $N_k, k\in\{1,2,3,\cdots,15\}$, $\forall(x,y)\in[0;+\infty[\times]-\infty;+\infty[$, $\overrightarrow{\nabla}\cdot\overrightarrow{H}\le 0$. We deduce that $\displaystyle\int_0^{+\infty}\overrightarrow{\nabla}\cdot\overrightarrow{H}dx \le 0$ $\forall y\in]-\infty;+\infty[$. N is bounded on $[0;+\infty[\times]-\infty;+\infty[$ as the limit of a sequence of functions bounded on $[0;+\infty[\times]-\infty;+\infty[$ so $\displaystyle\int_0^{+\infty}\overrightarrow{\nabla}\cdot\overrightarrow{H}dx$ exists and is finite, negative or zero and is a function of y. Let $J_k=[x_k,x_{k+1}[$ and $\varphi(J_k) = -\displaystyle\int_{x_k}^{x_{k+1}}\overrightarrow{\nabla}\cdot\overrightarrow{H}dx = \varphi_k(y)$ a function.

As $\overrightarrow{\nabla}.\overrightarrow{H}\le 0$, $\varphi(J_k)$ is a positive or zero function. Thus the sequence of function $(\varphi(J_k))_{k\in\mathbb{N}}$ is a sequence of positive functions(of positive functions of y) and the series of function of y, $\displaystyle\sum_k \varphi(J_k)$ is a series of positive functions. Let

$S_k = \sum_{n=0}^{k} \varphi(J_n)(y)$ its partial sums. Let S be the function defined from $]-\infty;+\infty[$ to $\mathbb{R}^+$ which to $y \longmapsto S(y) = \lim_{k\to+\infty} \sum_{n=0}^{k} \varphi(J_n)(y) = \sum_{n=0}^{+\infty} \varphi(J_n)(y)$.

As $J_k \cap J_l = \emptyset$ $\forall y \in]-\infty;+\infty[$ $\forall k, l \in \mathbb{N}, \sum_{n=0}^{+\infty} \varphi(J_n)(y) \leq -\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dx$. or $-\int_0^{+\infty} \overrightarrow{\nabla} \cdot \overrightarrow{H} dx$ exist and is finite then $S(y)$ exists and is finite. then the sequence of partial sums $(S_k)_{k\in\mathbb{N}}$ admits a finite limit so it converges to S on $]-\infty,+\infty[$. Thus the series of functions defined on $]-\infty;+\infty[$, $\sum_k \varphi(J_k)(y)$, simply converge to S. $\forall y \in]-\infty;+\infty[$ the series with general term $\varphi(J_k)$, $\sum_k \varphi(J_k)(y)$, is a convergent series with positive terms so its general term $\varphi(J_k)$ tends to zero when k tends to $+\infty$. But there exists $z_k \in J_k$ such that $\varphi_k(x) = -\int_{x_k}^{x_{k+1}} \overrightarrow{\nabla} \cdot \overrightarrow{H}(x,y)\, dx = -(x_{k+1} - x_k) \overrightarrow{\nabla} \cdot \overrightarrow{H}(z_k, y)$. The sequence $(z_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive real numbers tending towards $+\infty$. Let $u_k = x_{k+1} - x_k$. If the sequence $(u_k)_{k\in\mathbb{N}}$ converges to zero when k tends to $+\infty$ then $\forall \varepsilon > 0$, $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}$, $k > k_0 \Longrightarrow |x_{k+1} - x_k| < \varepsilon$. Which means that the sequence $(x_k)_{k\in\mathbb{N}}$ is from cauchy. Which is contradictory to the fact that it is divergent. Consequently, $\forall y \in]-\infty;+\infty[, \varphi_k(y)$ tends to zero when k tends to $+\infty$ if and only if $\overrightarrow{\nabla} \cdot \overrightarrow{H}(z_k, y)$ tends to zero as k tends to $+\infty$. But

$$\begin{aligned}
&\overrightarrow{\nabla} \cdot \overrightarrow{H}(z_k, y) \\
= \;& \sqrt{2}cs \ln\left(\frac{N_2(z_k,y)\,N_4(z_k,y)}{N_1(z_k,y)\,N_8(z_k,y)}\right)\left(1 - \frac{N_2(z_k,y)\,N_4(z_k,y)}{N_1(z_k,y)\,N_8(z_k,y)}\right) N_1(z_k,y)\,N_8(z_k,y) \\
+ \;& \sqrt{2}cs \ln\left(\frac{N_2(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_9(z_k,y)}\right)\left(1 - \frac{N_2(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_9(z_k,y)}\right) N_1(z_k,y)\,N_9(z_k,y) \\
+ \;& \sqrt{2}cs \ln\left(\frac{N_3(z_k,y)\,N_4(z_k,y)}{N_1(z_k,y)\,N_{12}(z_k,y)}\right)\left(1 - \frac{N_3(z_k,y)\,N_4(z_k,y)}{N_1(z_k,y)\,N_{12}(z_k,y)}\right) N_1(z_k,y)\,N_{12}(z_k,y) \\
+ \;& \sqrt{2}cs \ln\left(\frac{N_3(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{13}(z_k,y)}\right)\left(1 - \frac{N_3(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{13}(z_k,y)}\right) N_1(z_k,y)\,N_{13}(z_k,y) \\
+ \;& \sqrt{2}cs \ln\left(\frac{N_5(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{10}(z_k,y)}\right)\left(1 - \frac{N_5(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{10}(z_k,y)}\right) N_1(z_k,y)\,N_{10}(z_k,y) \\
+ \;& \sqrt{2}cs \ln\left(\frac{N_4(z_k,y)\,N_5(z_k,y)}{N_1(z_k,y)\,N_{11}(z_k,y)}\right)\left(1 - \frac{N_4(z_k,y)\,N_5(z_k,y)}{N_1(z_k,y)\,N_{11}(z_k,y)}\right) N_1(z_k,y)\,N_{11}(z_k,y)
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{2}cs\ln\left(\frac{N_4(x,z_k)\,N_6(z_k,y)}{N_1(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_4(z_k,y)\,N_6(z_k,y)}{N_1(z_k,y)\,N_{15}(z_k,y)}\right)N_1(z_k,y)\,N_{15}(z_k,y)\\
&+ \sqrt{2}cs\ln\left(\frac{N_6(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{14}(z_k,y)}\right)\left(1-\frac{N_6(z_k,y)\,N_7(z_k,y)}{N_1(z_k,y)\,N_{14}(z_k,y)}\right)N_1(z_k,y)\,N_{14}(z_k,y)\\
&+ 2cs\ln\left(\frac{N_2(z_k,y)\,N_5(z_k,y)}{N_3(z_k,y)\,N_6(x,z_k)}\right)\left(1-\frac{N_2(x,z_k)\,N_5(z_k,y)}{N_3(z_k,y)\,N_6(z_k,y)}\right)N_3(z_k,y)\,N_6(z_k,y)\\
&+ 2cs\ln\left(\frac{N_2(z_k,y)\,N_5(z_k,y)}{N_4(z_k,y)\,N_7(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_5(z_k,y)}{N_4(z_k,y)\,N_7(z_k,y)}\right)N_4(z_k,y)\,N_7(z_k,y)\\
&+ 2cs\ln\left(\frac{N_3(z_k,y)\,N_6(z_k,y)}{N_4(z_k,y)\,N_7(z_k,y)}\right)\left(1-\frac{N_3(z_k,y)\,N_6(z_k,y)}{N_4(z_k,y)\,N_7(z_k,y)}\right)N_4(z_k,y)\,N_7(z_k,y)\\
&+ 2cs\ln\left(\frac{N_8(z_k,y)\,N_{11}(z_k,y)}{N_{12}(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{11}(z_k,y)}{N_{12}(z_k,y)\,N_{15}(z_k,y)}\right)N_{12}(z_k,y)\,N_{15}(z_k,y)\\
&+ 2cs\ln\left(\frac{N_9(z_k,y)\,N_{10}(z_k,y)}{N_{13}(z_k,y)\,N_{14}(z_k,y)}\right)\left(1-\frac{N_9(z_k,y)\,N_{10}(x,z_k)}{N_{13}(x,z_k)\,N_{14}(x,z_k)}\right)N_{13}(z_k,y)\,N_{14}(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_2(z_k,y)\,N_{12}(z_k,y)}{N_3(z_k,y)\,N_8(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{12}(z_k,y)}{N_3(z_k,y)\,N_8(z_k,y)}\right)N_3(z_k,y)\,N_8(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_2(z_k,y)\,N_{13}(z_k,y)}{N_3(z_k,y)\,N_9(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{13}(z_k,y)}{N_3(z_k,y)\,N_9(z_k,y)}\right)N_3(z_k,y)\,N_9(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_2(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_8(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_8(z_k,y)}\right)N_6(z_k,y)\,N_8(x,z_k)\\
&+ \sqrt{3}cs\ln\left(\frac{N_2(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_9(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_9(z_k,y)}\right)N_6(z_k,y)\,N_9(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N(z_k,y)\,N_{12}(z_k,y)}{N_3(z_k,y)\,N_{11}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_{12}(z_k,y)}{N_3(z_k,y)\,N_{11}(z_k,y)}\right)N_3(z_k,y)\,N_{11}(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_5(z_k,y)\,N_{13}(z_k,y)}{N_3(z_k,y)\,N_{10}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_{13}(z_k,y)}{N_3(z_k,y)\,N_{10}(z_k,y)}\right)N_3(z_k,y)\,N_{10}(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_5(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_{10}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_{10}(z_k,y)}\right)N_6(z_k,y)\,N_{10}(z_k,y)\\
&+ \sqrt{3}cs\ln\left(\frac{N_5(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_{11}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_{11}(z_k,y)}\right)N_6(z_k,y)\,N_{11}(z_k,y)\\
&+ \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_3(z_k,y)\,N_{14}(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_3(z_k,y)\,N_{14}(z_k,y)}\right)N_3(z_k,y)\,N_{14}(z_k,y)\\
&+ \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_3(x,z_k)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_3(z_k,y)\,N_{15}(z_k,y)}\right)N_3(z_k,y)\,N_{15}(z_k,y)\\
&+ \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_5(z_k,y)\,N_8(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_5(z_k,y)\,N_8(z_k,y)}\right)N_5(z_k,y)\,N_8(z_k,y)\\
&+ \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_5(z_k,y)\,N_9(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_5(z_k,y)\,N_9(z_k,y)}\right)N_5(z_k,y)\,N_9(z_k,y)\\
&+ \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{11}(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)N_6(z_k,y)\,N_{12}(z_k,y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sqrt{5}cs\ln\left(\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)\left(1-\frac{N_2(z_k,y)\,N_{10}(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)N_6(z_k,y)\,N_{13}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)\left(1-\frac{N_3(z_k,y)\,N_{14}(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)N_6(z_k,y)\,N_{13}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(z_k,y)\,N_{14}(z_k,y)}{N_5(z_k,y)\,N_9(z_k,y)}\right)\left(1-\frac{N_3(z_k,y)\,N_{14}(z_k,y)}{N_5(z_k,y)\,N_9(z_k,y)}\right)N_5(z_k,y)\,N_9(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(z_k,y)\,N_{15}(z_k,y)}{N_5(z_k,y)\,N_8(z_k,y)}\right)\left(1-\frac{N_3(z_k,y)\,N_{15}(z_k,y)}{N_5(z_k,y)\,N_8(z_k,y)}\right)N_5(z_k,y)\,N_8(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_3(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)\left(1-\frac{N_3(z_k,y)\,N_{15}(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)N_6(z_k,y)\,N_{12}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_5(z_k,y)\,N_8(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_8(z_k,y)}{N_6(z_k,y)\,N_{12}(z_k,y)}\right)N_6(z_k,y)\,N_{12}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_5(z_k,y)\,N_9(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)\left(1-\frac{N_5(z_k,y)\,N_9(z_k,y)}{N_6(z_k,y)\,N_{13}(z_k,y)}\right)N_6(z_k,y)\,N_{13}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_7(z_k,y)\,N_8(z_k,y)}{N_4(z_k,y)\,N_9(z_k,y)}\right)\left(1-\frac{N_7(z_k,y)\,N_8(z_k,y)}{N_4(z_k,y)\,N_9(z_k,y)}\right)N_4(z_k,y)\,N_9(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_4(z_k,y)\,N_{10}(z_k,y)}{N_7(z_k,y)\,N_{11}(z_k,y)}\right)\left(1-\frac{N_4(z_k,y)\,N_{10}(z_k,y)}{N_7(z_k,y)\,N_{11}(z_k,y)}\right)N_7(z_k,y)\,N_{11}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_7(z_k,y)\,N_{12}(z_k,y)}{N_4(z_k,y)\,N_{13}(z_k,y)}\right)\left(1-\frac{N_7(z_k,y)\,N_{12}(z_k,y)}{N_4(z_k,y)\,N_{13}(z_k,y)}\right)N_4(z_k,y)\,N_{13}(z_k,y)\\
&+ \quad \sqrt{5}cs\ln\left(\frac{N_4(z_k,y)\,N_{14}(z_k,y)}{N_7(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_4(z_k,y)\,N_{14}(z_k,y)}{N_7(z_k,y)\,N_{15}(z_k,y)}\right)N_7(z_k,y)\,N_{15}(z_k,y)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_8(z_k,y)\,N_{13}(z_k,y)}{N_9(z_k,y)\,N_{12}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{13}(z_k,y)}{N_9(z_k,y)\,N_{12}(z_k,y)}\right)N_9(z_k,y)\,N_{12}(z_k,y)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_8(z_k,y)\,N_{14}(z_k,y)}{N_9(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{14}(z_k,y)}{N_9(z_k,y)\,N_{15}(z_k,y)}\right)N_9(z_k,y)\,N_{15}(z_k,y)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_{10}(z_k,y)\,N_{12}(z_k,y)}{N_{11}(z_k,y)\,N_{13}(z_k,y)}\right)\left(1-\frac{N_{10}(z_k,y)\,N_{12}(z_k,y)}{N_{11}(z_k,y)\,N_{13}(z_k,y)}\right)N_{11}(z_k,y)\,N_{13}(z_k,y)\\
&+ \quad \sqrt{6}cs\ln\left(\frac{N_{11}(z_k,y)\,N_{14}(z_k,y)}{N_{15}(z_k,y)\,N_{10}(z_k,y)}\right)\left(1-\frac{N_{11}(z_k,y)\,N_{14}(z_k,y)}{N_{15}(z_k,y)\,N_{10}(z_k,y)}\right)N_{15}(z_k,y)\,N_{10}(z_k,y)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_9(z_k,y)\,N_{11}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_9(z_k,y)\,N_{11}(z_k,y)}\right)N_9(z_k,y)\,N_{11}(z_k,y)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_{12}(z_k,y)\,N_{14}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_{12}(z_k,y)\,N_{14}(z_k,y)}\right)N_{12}(x,z_k)\,N_{14}(x,z_k)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_{13}(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_8(z_k,y)\,N_{10}(z_k,y)}{N_{13}(z_k,y)\,N_{15}(z_k,y)}\right)N_{13}(z_k,y)\,N_{15}(z_k,y)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_9(z_k,y)\,N_{11}(z_k,y)}{N_{12}(z_k,y)\,N_{14}(z_k,y)}\right)\left(1-\frac{N_9(z_k,y)\,N_{11}(z_k,y)}{N_{12}(z_k,y)\,N_{14}(z_k,y)}\right)N_{12}(z_k,y)\,N_{14}(z_k,y)\\
&+ \quad 2\sqrt{2}cs\ln\left(\frac{N_9(z_k,y)\,N_{11}(z_k,y)}{N_{13}(z_k,y)\,N_{15}(z_k,y)}\right)\left(1-\frac{N_9(z_k,y)\,N_{11}(z_k,y)}{N_{13}(z_k,y)\,N_{15}(z_k,y)}\right)N_{13}(z_k,y)\,N_{15}(z_k,y)
\end{aligned}
$$

$$+ \quad 2\sqrt{2}cs\ln\left(\frac{N_{12}\left(z_k,y\right)N_{14}\left(z_k,y\right)}{N_{13}\left(z_k,y\right)N_{15}\left(z_k,y\right)}\right)\left(1-\frac{N_{12}\left(z_k,y\right)N_{14}\left(z_k,y\right)}{N_{13}\left(z_k,y\right)N_{15}\left(z_k,y\right)}\right)N_{13}\left(z_k,y\right)N_{15}\left(z_k,y\right).$$

So, if $N_\alpha^M\left(y\right)=\lim\limits_{k\longrightarrow+\infty}N_\alpha\left(\left(z_k,y\right)\right),\alpha\in\{1,2,3,\ldots,15\}$ then $\lim\limits_{k\to+\infty}\overrightarrow{\nabla}\cdot\overrightarrow{H}\left(z_k,y\right)=0$ is only possible for positive $N_\alpha^M\left(x\right),\alpha=1,2,3,\ldots,15$ if

$$\begin{aligned}
&N_3^M(y)N_4^M(y)=N_1^M(y)N_{12}^M(y),\ N_3^M(y)N_7^M(y)=N_1^M(y)N_{13}^M(y),\\
&N_5^M(y)N_7^M(y)=N_1^M(y)N_{10}^M(y),\ N_4^M(x)N_5^M(y)=N_1^M(y)N_{11}^M(y),\\
&N_4^M(y)N_6^M(y)=N_1^M(y)N_{15}^M(y),\ N_6^M(y)N_7^M(y)=N_1^M(y)N_{14}^M(y),\\
&N_2^M(y)N_5^M(y)=N_3^M(y)N_6^M(y),\ N_2^M(y)N_5^M(y)=N_4^M(y)N_7^M(y),\\
&N_3^M(y)N_6^M(y)=N_4^M(y)N_7^M(y),\ N_8^M(y)N_{11}^M(y)=N_{12}^M(y)N_{15}^M(y),\\
&N_9^M(y)N_{10}^M(y)=N_{13}^M(y)N_{14}^M(y),\ N_2^M(y)N_4^M(y)=N_1^M(y)N_8^M(y),\\
&N_2^M(y)N_7^M(y)=N_1^M(y)N_9^M(y),\ N_2^M(y)N_{12}^M(y)=N_3^M(y)N_8^M(y),\\
&N_2^M(y)N_{13}^M(y)=N_3^M(y)N_9^M(y),\ N_2^M(y)N_{15}^M(y)=N_6^M(y)N_8^M(y),\\
&N_2^M(y)N_{14}^M(y)=N_6^M(y)N_9^M(y),\ N_5^M(y)N_{12}^M(y)=N_3^M(y)N_{11}^M(y),\\
&N_5^M(y)N_{13}^M(y)=N_3^M(y)N_{10}^M(y),\ N_5^M(y)N_{14}^M(y)=N_6^M(y)N_{10}^M(y),\\
&N_5^M(y)N_{15}^M(y)=N_6^M(y)N_{11}^M(y),\ N_8^M(y)N_{13}^M(y)=N_9^M(y)N_{12}^M(y),\\
&N_8^M(y)N_{14}^M(y)=N_9^M(y)N_{15}^M(y),\ N_{10}^M(y)N_{12}^M(y)=N_{11}^M(y)N_{13}^M(y),\\
&N_{11}^M(y)N_{14}^M(y)=N_{10}^M(y)N_{15}^M(y),\ N_8^M(y)N_{10}^M(y)=N_9^M(y)N_{11}^M(y),\\
&N_8^M(y)N_{10}^M(y)=N_{12}^M(y)N_{14}^M(y),\ N_8^M(y)N_{10}^M(y)=N_{13}^M(y)N_{15}^M(y),\\
&N_9^M(y)N_{11}^M(y)=N_{12}^M(y)N_{14}^M(y),\ N_9^M(y)N_{11}^M(y)=N_{13}^M(y)N_{15}^M(y),\\
&N_{12}^M(y)N_{14}^M(y)=N_{13}^M(y)N_{15}^M(y),\ N_2^M(y)N_{10}^M(y)=N_3^M(y)N_{14}^M(y),\\
&N_2^M(y)N_{11}^M(y)=N_3^M(y)N_{15}^M(y),\ N_2^M(y)N_{11}^M(y)=N_5^M(y)N_8^M(y),\\
&N_2^M(y)N_{10}^M(y)=N_5^M(y)N_9^M(y),\ N_2^M(y)N_{11}^M(y)=N_6^M(y)N_{12}^M(x),\\
&N_2^M(y)N_{10}^M(y)=N_6^M(y)N_{13}^M(y),\ N_3^M(y)N_{14}^M(y)=N_6^M(y)N_{13}^M(y),\\
&N_3^M(y)N_{14}^M(y)=N_5^M(y)N_9^M(y),\ N_3^M(y)N_{15}^M(y)=N_5^M(y)N_8^M(y),\\
&N_3^M(y)N_{15}^M(y)=N_6^M(y)N_{12}^M(y),\ N_5^M(y)N_8^M(y)=N_6^M(y)N_{12}^M(y),\\
&N_5^M(y)N_9^M(y)=N_6^M(y)N_{13}^M(y),\ N_7^M(y)N_8^M(y)=N_4^M(y)N_9^M(y),\\
&N_4^M(y)N_{10}^M(y)=N_7^M(y)N_{11}^M(y),\ N_7^M(y)N_{12}^M(y)=N_4^M(y)N_{13}^M(y),\\
&N_4^M(y)N_{14}^M(y)=N_7^M(y)N_{15}^M(y),
\end{aligned}$$

i.e N^M is a Maxwellian equilibrium state. □

4. Conclusion

we extend in a strip the solutions of the boundary value problem for the fifteen velocity discrete model proven in a rectangle. Afterwards we extend in a half space the solutions obtained in the strip.we also show that these solutions tend towards a Maxwellian state at infinity.

Appendix

The kinetic equations of the model.

$$\begin{aligned}
\frac{\partial N_1}{\partial t'} &= \sqrt{2}cs[(N_2+N_3+N_5+N_6)(N_4+N_7)-(N_8+N_9+N_{10}+N_{11}+N_{12}+N_{13})N_1] \\
&- \sqrt{2}cs[(N_{14}+N_{15})N_1] \\
\frac{\partial N_2}{\partial t'}+c\frac{\partial N_2}{\partial x'} &= \sqrt{3}cs\left[(N_3+N_6)(N_8+N_9)-N_2(N_{12}+N_{13}+N_{14}+N_{15})\right] \\
&+ \sqrt{5}cs\left[N_3(N_{14}+N_{15})+N_5(N_8+N_9)+N_6(N_{12}+N_{13})-3N_1(N_{10}+N_{11})\right] \\
&\ \sqrt{2}cs\left[N_1(N_8+N_9)-N_2(N_4+N_7)\right]+2cs\left(N_3N_6+N_4N_7-2N_2N_5\right) \\
\frac{\partial N_3}{\partial t'}+c\frac{\partial N_3}{\partial y'} &= \sqrt{3}cs\left[(N_3+N_6)(N_{12}+N_{13})-N_3(N_8+N_9+N_{10}+N_{11})\right] \\
&+ \sqrt{5}cs\left[N_2(N_{10}+N_{11})+N_5(N_8+N_9)+N_6(N_{12}+N_{13})-3N_3(N_{14}+N_{15})\right] \\
&+ \sqrt{2}cs\left[N_1(N_{12}+N_{13})-N_3(N_4+N_7)\right]+2cs\left(N_2N_5+N_4N_7-N_3N_6\right) \\
\frac{\partial N_4}{\partial t'}+c\frac{\partial N_4}{\partial z'} &= \sqrt{2}cs[N_1(N_8+N_{11}+N_{12}+N_{15})-N_4(N_2+N_3+N_5+N_6)]+2cs(N_2N_5 \\
&+ N_3N_6-2N_4N_7)+\sqrt{5}cs[N_7(N_8+N_{11}+N_{12}+N_{15})-N_4(N_9+N_{10}+N_{13}+N_{14})] \\
\frac{\partial N_5}{\partial t'}-c\frac{\partial N_5}{\partial x'} &= \sqrt{3}cs\left[(N_3+N_6)(N_{10}+N_{11})-N_5(N_{12}+N_{13}+N_{14}+N_{15})\right] \\
&+ \sqrt{5}cs\left[N_2(N_{10}+N_{11})+N_3(N_{14}+N_{15})+N_6(N_{12}+N_{13})-3N_5(N_8+N_9)\right] \\
&+ \sqrt{2}cs\left[N_1(N_{10}+N_{11})-N_5(N_4+N_7)\right]+2cs\left(N_3N_6+N_4N_7-2N_2N_5\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial N_6}{\partial t'}-c\frac{\partial N_6}{\partial y'} \\
=\ &\sqrt{3}cs\left[(N_2+N_5)(N_{14}+N_{15})-N_6(N_8+N_9+N_{10}+N_{11})\right] \\
+\ &\sqrt{5}cs\left[N_2(N_{10}+N_{11})+N_3(N_{14}+N_{15})+N_5(N_8+N_9)-3N_6(N_{12}+N_{13})\right] \\
+\ &\sqrt{2}cs\left[N_1(N_{14}+N_{15})-N_6(N_4+N_7)\right]+2cs\left(N_2N_5+N_4N_7-2N_3N_6\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial N_7}{\partial t'}-c\frac{\partial N_7}{\partial z'} \\
=\ &\sqrt{2}cs[N_1(N_9+N_{10}+N_{13}+N_{14})-N_7(N_2+N_3+N_5+N_6)]+2cs(N_2N_5 \\
+\ &N_3N_6-2N_4N_7)+\sqrt{5}cs[N_4(N_9+N_{10}+N_{13}+N_{14})-N_7(N_8+N_{11}+N_{12}+N_{15})]
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial N_8}{\partial t'}+c\frac{\partial N_8}{\partial x'}+c\frac{\partial N_8}{\partial z'} \\
=\ &2cs(N_{12}N_{15}-N_8N_{11})+\sqrt{5}cs(N_2N_{11}+N_3N_{15}+N_4N_9+N_6N_{12} \\
-\ &3N_5N_8-N_7N_8)+\sqrt{3}cs[N_2(N_{12}+N_{15})-N_8(N_3+N_6)]+\sqrt{6}cs[N_9(N_{12}+N_{15})-N_8N_{13}] \\
-\ &\sqrt{6}csN_8N_{14}+2\sqrt{2}cs(N_9N_{11}+N_{12}N_{14}+N_{13}N_{15}-3N_8N_{10})+\sqrt{2}cs(N_2N_4-N_1N_8)
\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial N_9}{\partial t'} + c\frac{\partial N_9}{\partial x'} - c\frac{\partial N_9}{\partial z'} \\
= & \sqrt{3}cs\left[N_2\left(N_{13}+N_{14}\right) - N_8\left(N_3+N_6\right)\right] + 2cs\left(N_{13}N_{14} - N_9N_{10}\right) \\
+ & \sqrt{2}cs\left(N_2N_4 - N_1N_8\right) + \sqrt{5}cs\left(N_2N_{10} + N_3N_{14} + N_6N_{13} + N_7N_8 - N_4N_9 - 3N_5N_9\right) \\
+ & \sqrt{6}cs\left[N_8\left(N_{13}+N_{14}\right) - N_9\left(N_{12}+N_{15}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_{12}N_{14} + N_{13}N_{15} - 3N_9N_{11}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{10}}{\partial t'} - c\frac{\partial N_{10}}{\partial x'} - c\frac{\partial N_{10}}{\partial z'} \\
= & \sqrt{3}cs\left[N_5\left(N_{13}+N_{14}\right) - N_{10}\left(N_3+N_6\right)\right] + 2cs\left(N_{13}N_{14} - N_9N_{10}\right) \\
+ & \sqrt{2}cs\left(N_5N_7 - N_1N_{10}\right) + \sqrt{5}cs\left(N_3N_{14} + N_5N_9 + N_6N_{13} + N_7N_{11} - N_4N_{10} - 3N_2N_{10}\right) \\
+ & \sqrt{6}cs\left[N_{11}\left(N_{13}+N_{14}\right) - N_{10}\left(N_{12}+N_{15}\right)\right] + 2\sqrt{2}cs\left(N_9N_{11} + N_{12}N_{14} + N_{13}N_{15} - 3N_8N_{10}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{11}}{\partial t'} - c\frac{\partial N_{11}}{\partial x'} + c\frac{\partial N_{11}}{\partial z'} \\
= & \sqrt{3}cs\left[N_5\left(N_{12}+N_{15}\right) - N_{11}\left(N_3+N_6\right)\right] + 2cs\left(N_{12}N_{15} - N_8N_{11}\right) \\
+ & \sqrt{2}cs\left(N_4N_5 - N_1N_{11}\right) + \sqrt{5}cs\left(N_3N_{13} + N_5N_8 + N_6N_{12} + N_4N_{10} - N_7N_{11} - 3N_2N_{11}\right) \\
+ & \sqrt{6}cs\left[N_{10}\left(N_{12}+N_{15}\right) - N_{11}\left(N_{13}+N_{14}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_{12}N_{14} + N_{13}N_{15} - 3N_9N_{11}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{12}}{\partial t'} + c\frac{\partial N_{12}}{\partial y'} + c\frac{\partial N_{12}}{\partial z'} \\
= & \sqrt{3}cs\left[N_3\left(N_8+N_{11}\right) - N_{12}\left(N_2+N_5\right)\right] + 2cs\left(N_8N_{11} - N_{12}N_{15}\right) \\
+ & \sqrt{2}cs\left(N_3N_4 - N_1N_{12}\right) + \sqrt{5}cs\left(N_2N_{11} + N_3N_{15} + N_4N_{13} + N_5N_8 - N_7N_{12} - 3N_6N_{12}\right) \\
+ & \sqrt{6}cs\left[N_{13}\left(N_8+N_{11}\right) - N_{12}\left(N_9+N_{10}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_9N_{11} + N_{13}N_{15} - 3N_{12}N_{14}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{13}}{\partial t'} + c\frac{\partial N_{13}}{\partial y'} - c\frac{\partial N_{13}}{\partial z'} \\
= & \sqrt{3}cs\left[N_3\left(N_9+N_{10}\right) - N_{31}\left(N_2+N_5\right)\right] + 2cs\left(N_9N_{10} - N_{13}N_{14}\right) \\
+ & \sqrt{2}cs\left(N_3N_7 - N_1N_{13}\right) + \sqrt{5}cs\left(N_2N_{110} + N_3N_{14} + N_5N_9 + N_7N_{12} - N_4N_{13} - 3N_6N_{13}\right) \\
+ & \sqrt{6}cs\left[N_{12}\left(N_9+N_{10}\right) - N_{13}\left(N_8+N_{11}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_9N_{11} + N_{12}N_{14} - 3N_{13}N_{15}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{14}}{\partial t'} - c\frac{\partial N_{14}}{\partial y'} - c\frac{\partial N_{14}}{\partial z'} \\
= & \sqrt{3}cs\left[N_6\left(N_9+N_{10}\right) - N_{14}\left(N_2+N_5\right)\right] + 2cs\left(N_9N_{10} - N_{13}N_{14}\right) \\
+ & \sqrt{2}cs\left(N_6N_7 - N_1N_{14}\right) + \sqrt{5}cs\left(N_2N_{10} + N_5N_9 + N_6N_{13} + N_7N_{15} - N_4N_{14} - 3N_3N_{14}\right) \\
+ & \sqrt{6}cs\left[N_{15}\left(N_9+N_{10}\right) - N_{14}\left(N_8+N_{11}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_9N_{11} + N_{13}N_{15} - 3N_{12}N_{14}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial N_{15}}{\partial t'} - c\frac{\partial N_{15}}{\partial y'} + c\frac{\partial N_{15}}{\partial z'} \\
= & \sqrt{3}cs\left[N_6\left(N_8+N_{11}\right) - N_{15}\left(N_2+N_5\right)\right] + 2cs\left(N_8N_{11} - N_{12}N_{15}\right) \\
+ & \sqrt{2}cs\left(N_4N_6 - N_1N_{15}\right) + \sqrt{5}cs\left(N_2N_{11} + N_5N_8 + N_6N_{12} + N_4N_{14} - N_7N_{15} - 3N_3N_{15}\right) \\
+ & \sqrt{6}cs\left[N_{14}\left(N_8+N_{11}\right) - N_{15}\left(N_9+N_{10}\right)\right] + 2\sqrt{2}cs\left(N_8N_{10} + N_9N_{11} + N_{12}N_{14} - 3N_{13}N_{15}\right)
\end{aligned}
$$

CONFLICT OF INTEREST

The author declares no competing interests.

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