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A NOTE ON A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The object of this paper is to obtain sharp results involving coefficients bounds, growth and distortion properties for a subclass of the class of close-to-convex functions. We also consider the Fekete-Szegö problem for the same class.

1. Introduction and definitions

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$.

Let C denote the class of convex functions [1], $f(z) \in C$ if and only if for $z \in E$,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0\,.$$

Let S^* denote the class of starlike functions [2]:

$$f(z) \in S^* \text{ if and only if for } z \in E, \Re\frac{zf'(z)}{f(z)} > 0\,.$$

A function $f(z)$ analytic in E is said to be close-to-convex in E, if there exists a function $g(z) \in S^*$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0\,.$$

The class of such functions is denoted by K , see [3]. The classes S, K, S^* and C are related by the proper inclusions

$$C \subset S^* \subset K \subset S\,.$$

Now we will consider a class $\widetilde{K}$ defined as follows.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f(z) \in \widetilde{K}$ if and only if there exists a function $g(z) \in C$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0\,. \tag{1.1}$$

Since $C \subset S^*$, it follows that $\widetilde{K} \subset K$ and so, the functions in $\widetilde{K}$ are univalent.

Let P be the class of functions $h(z)$ given by $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, which are analytic and have positive real part in E.

Let Ω be the class of functions ω analytic in E such that $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in E$.

2. Known results

Theorem 2.1. ([4]) *If $g(z) \in C$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then:*

(i) $|b_n| \leq 1 \qquad (n = 2, 3, \ldots)$,

(ii) $\left|b_3 - \mu b_2^2\right| \leq \max\left(\frac{1}{3}, |1-\mu|\right)$,

(iii) $\left|b_3 - \mu b_2^2\right| \leq \frac{1}{3}\left(1 - |b_2|^2\right)$.

Theorem 2.2. ([5]) *Let $h(z) \in P$, with $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then*

(i) $|c_n| \leq 2 \qquad (n = 1, 2, \ldots)$,

(ii) $\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}$.

Equality holds when $h(z) = \frac{1+z}{1-z}$.

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Theorem 2.3. ([4]) *Let* $g(z) \in C$, *with* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. *Then, for* $z = re^{i\theta} \in E$,

$$\begin{aligned} \frac{r}{1+r} &\le |g(z)| \le \frac{r}{1-r}, \\ \frac{1}{(1+r)^2} &\le |g'(z)| \le \frac{1}{(1-r)^2}, \\ \frac{1}{1+r} &\le \left|\frac{zg'(z)}{g(z)}\right| \le \frac{1}{1-r}. \end{aligned}$$

Equality holds if and only if $g(z) = \frac{z}{(1-\varepsilon z)}$, $|\varepsilon| = 1$.

Theorem 2.4. ([2]) *Function* $h(z) \in P$ *if, and only if*

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)}, \qquad z \in E,$$

where $\omega \in \Omega$.

Theorem 2.5. ([6]) *If* $h(z) \in P$, *then for* $z = re^{i\theta} \in E$

$$\begin{aligned} &\text{(i)} & \frac{1-r}{1+r} &\le |h(z)| \le \frac{1+r}{1-r}, \\ &\text{(ii)} & \left|\frac{zh'(z)}{h(z)}\right| &\le \frac{2r}{1-r^2}, \\ &\text{(iii)} & |h'(z)| &\le \frac{2\Re\, h(z)}{1-r^2}. \end{aligned}$$

Equality is attained when $h(z) = \frac{1+\varepsilon z}{1-\varepsilon z}$, $|\varepsilon| = 1$.

3. Some of the basic properties of functions in $\widetilde{K}$

Theorem 3.1. *Let* $f(z) \in \widetilde{K}$. *Then for* $z = re^{i\theta} \in E$,

$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2},$$

$$-\ln(1+r) + \frac{2r}{1+r} \le |f(z)| \le \ln(1-r) + \frac{2r}{1-r}.$$

Each inequality is sharp for $f_0(z)$ *defined by*

$$f_0(z) = \overline{x}\log(1-xz) + \frac{zx}{1-xz}, \quad \text{with } |x| = 1. \tag{3.1}$$

Proof. Since $f(z) \in \widetilde{K}$, (1.1) gives

$$|zf'(z)| = |g(z)h(z)|,$$

for $g(z) \in C$ and $h(z) \in P$. From Theorem 2.3 and Theorem 2.5 we have

$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2}. \tag{3.2}$$

Integrating (3.2) along the straight line segment from the origin to $z = re^{i\theta}$ and using the right inequality of (3.2) we obtain

$$|f(z)| \le \int_0^z |f'(z)||dz| \le \int_0^r \frac{1+t}{(1-t)^2} dt$$

$$= \ln(1-r) + \frac{2r}{1-r},$$

which gives the upper bound for $|f(z)|$.

In order to obtain the lower bound for $|f(z)|$, we proceed as follows: let z_1 be such that $|z_1| = r$ and satisfies $|f(z_1)| \le |f(z)|$ for all z with $|z| = r$. Writing $\omega = f(z)$, it follows that the line segment λ from $\omega = 0$ to $\omega = f(z)$ lies entirely in the image of f.

Let Λ be the pre-image of λ. Then

$$|f(z)| \ge |f(z_1)| = \int_\lambda |d\omega| = \int_\Lambda \left|\frac{d\omega}{dz}\right| |dz|$$

$$\ge \int_0^r \frac{1-t}{(1+t)^2} dt = -\ln(1+r) + \frac{2r}{1+r},$$

which is the required lower bound.

Equality is attained on choosing $g(z) = \frac{z}{1-xz}$ and $h(z) = \frac{1+xz}{1-xz}$ for $|x| = 1$ in the representation $zf'(z) = g(z)h(z)$. ☐

Theorem 3.2. *Let* $f(z) \in \widetilde{K}$, *with* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then for* $z \in E$,

$$|a_n| \le 2 - \frac{1}{n} \quad \text{for} \quad n \ge 2.$$

Equality is attained for $f_0(z)$ *defined in (3.1).*

Proof. It follows from (1.1) that we can write

$$zf'(z) = g(z)h(z) \tag{3.3}$$

for $g(z) \in C$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $h(z) \in P$, with $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$.

Equating the coefficients of z^n in (3.3), we have for $n \ge 2$,

$$na_n = b_n + \sum_{k=1}^{n-2} b_{n-k}c^k + c_{n-1}.$$

Thus for $n \ge 2$:

$$\begin{aligned} n|a_n| &\le |b_n| + \sum_{k=1}^{n-2} |b_{n-k}||c_k| + |c_{n-1}| \\ &\le 3 + 2\sum_{k=1}^{n-2} |b_{n-k}| \le 2n-1, \\ \text{i.e.} \quad & |a_n| \le 2 - \frac{1}{n}, \end{aligned}$$

using Theorem 2.1 and Theorem 2.2. ☐

We now consider the Fekete-Szegö problem for the class $\widetilde{K}$.

Theorem 3.3. *Let $f(z) \in \widetilde{K}$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{5}{3} - \frac{9}{4}\mu, \ \text{if } \mu \leq \frac{2}{9}, \\ \frac{2}{3} + \frac{1}{9\mu}, \ \text{if } \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\ \frac{5}{6}, \ \text{if } \frac{2}{3} \leq \mu \leq 1. \end{cases}$$

For each μ, there is a function in $\widetilde{K}$ such that equality holds.

Proof. In (1.1) Theorem 2.4 gives

$$h(z) = \frac{zf'(z)}{g(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

so that ω is analytic, $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for $z \in E$. Let $\omega(z)$ be given by:

$$\omega(z) = \sum_{n=1}^{\infty} \alpha_n z^n \quad \text{for} \quad z \in E.$$

Then Theorem 2.2 gives

$$|\alpha_1| \leq 1 \quad \text{and} \quad |\alpha_2| \leq 1 - |\alpha_1|^2,$$

see [5]. Now (3.3) gives

$$zf'(z) - g(z) = \{zf'(z) + g(z)\}\,\omega(z).$$

Equating the coefficients of z^2 and z^3 on both sides we obtain:

$$2a_2 = b_2 + 2\alpha_1,$$

and

$$3a_3 = b_3 + 2\alpha_2 + 2\alpha_1^2 + 2\alpha_1 b_2,$$

so that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \\ &+\frac{2}{3}\left(\alpha_2 + \left(1 - \frac{3}{2}\mu\right)\alpha_1^2\right) + \left(\frac{2}{3} - \mu\right)\alpha_1 b_2. \end{aligned} \tag{3.4}$$

We first consider the case $\frac{2}{9} \leq \mu \leq \frac{2}{3}$. Expression (3.4) gives:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \\ &\quad + \frac{2}{3}\left|\alpha_2 - \frac{1}{2}(3\mu - 2)\alpha_1^2\right| \\ &\quad + \frac{(2-3\mu)}{3}\left|\alpha_1 b^2\right| \\ &\leq \frac{4-3\mu}{12} + \frac{2}{3} - \mu|\alpha_1|^2 + \frac{(2-3\mu)}{3}|\alpha_1| = \Phi(t) \end{aligned}$$

say, with $t = |\alpha_1|$ where we have used Theorem 2.1 and Theorem 2.2 and the fact that $|b_2| \leq 1$ for $g(z) \in C$. Since the function $\Phi(t)$ attains its maximum at $t_0 = \frac{2-3\mu}{6\mu}$,

$$|a_3 - \mu a_2^2| \leq \Phi(t_0),$$

which proves the theorem if $\mu \leq \frac{2}{3}$. Choosing $\alpha_1 = \frac{2-3\mu}{6\mu}$, $\alpha_2 = 1 - \alpha_1^2$ and $b_1 = b_2 = 1$ in (3.4), shows that the result is sharp. We note that since $|\alpha_1| \leq 1$, we have $\mu \geq \frac{2}{9}$.

Next we suppose that $\mu \leq \frac{2}{9}$. Again, (3.4) gives:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{9}{2}\mu\left|a_3 - \frac{2}{9}a_2^2\right| + \left(1 - \frac{9}{2}\mu\right)|a_3| \\ &\leq \frac{9}{2}\mu \times \frac{7}{6} + \left(1 - \frac{9}{2}\mu\right)\frac{5}{3} = \frac{5}{3} - \frac{9}{4}\mu, \end{aligned}$$

where we have used the result already proved in case $\mu = \frac{2}{9}$, and the fact that for $f \in \widetilde{K}$, the inequality $|a_3| \leq \frac{5}{3}$ holds (Theorem 3.2). Equality is attained on choosing $b_2 = b_3 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 0$ in (3.4).

Now, suppose that $\frac{2}{3} \leq \mu \leq 1$. We deal first with the case $\mu = 1$. From (3.4), we have

$$a_3 - a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}b_2^2\right) + \frac{2}{3}\left(\alpha_2 - \frac{1}{2}\alpha_1^2\right) - \frac{1}{3}\alpha_1 b_2,$$

or

$$a_3 - a_2^2 = \frac{1}{3}\left(b_3 - b_2^2\right) + \frac{1}{12}b_2^2 + \frac{2}{3}\left(\alpha_2 - \frac{1}{2}\alpha_1^2\right) - \frac{\alpha_1 b_2}{3},$$

and so

$$|a_3 - a_2^2| \leq \frac{1}{3}\left|b_3 - b_2^2\right| + \frac{|b_2|^2}{12} + \frac{2}{3}\left|\alpha_2 - \frac{1}{2}\alpha_1^2\right| + \frac{|\alpha_1 b_2|}{3}.$$

Using Theorem 2.1 and Theorem 2.2, and the fact that $|\alpha_2| \leq 1 - |\alpha_1|^2$, we obtain:

$$\begin{aligned} |a_3 - a_2^2| &\leq \frac{1}{9}\left(1 - |b_2|^2\right) + \frac{|b_2|^2}{12} + \frac{2}{3} - \frac{|\alpha_1|^2}{3} + \frac{|a_1 b_2|}{3} \\ &= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{1}{3}\left(|\alpha_1| - \frac{|b_2|}{2}\right)^2 \leq \frac{7}{9} + \frac{|b_2|^2}{18} \\ &\leq \frac{5}{6}, \end{aligned}$$

since $|b_2| \leq 1$.

Next,

$$a_3 - \mu a_2^2 = (3\mu - 2)(a_3 - a_2^2) + 3(1-\mu)\left(a_3 - \frac{2}{3}a_2^2\right),$$

and the result follows at once on using the theorem already proved in the case $\mu = 1$ and $\mu = \frac{2}{3}$.

Equality is attained in this case when $b_2 = b_3 = 1$, $\alpha_2 = 1 - \alpha_1^2$ with

$$\alpha_1 = \frac{2-3\mu}{6\mu} \pm i\frac{\sqrt{(6\mu-4)}}{6\mu}.$$

□

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