

## SOME PROPERLY HEREDITARY BITOPOLOGICAL PROPERTIES

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**ABSTRACT.** In this paper we discuss some pairwise properly hereditary properties concerning pairwise separation axiom, and obtain several results related to these properties.

### 1. INTRODUCTION

The concept of bitopological spaces was initiated by Kelly ([5]). A bitopological space is an order triple  $(X, \tau_1, \tau_2)$  where  $X$  is a non-empty set and  $\tau_1, \tau_2$  are two topologies on  $X$ . Since then several mathematicians studied various properties in bitopological spaces and bitopological spaces turned to be an important field in general topology, Fletcher, P., Hoyle, H.B. and Patty, C.W. (1969) [3], Kim, Y.W. (1968) [7], Foran, A. and Hdeib, H. (1983) [4], Kilicman, A. and Salleh, Z. (2007) [6]. Several results were obtained in the above studies that generalize topological properties in bitopological spaces. Still pairwise properly hereditary properties are not investigated. In this paper we discuss some pairwise properly hereditary properties and try to obtain various results concerning their properties. Then  $\tau_i$ -closure of a set  $A$  will be denoted by  $cl_i A$

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## 2. SOME PROPERLY HEREDITARY BITOPOLOGICAL PROPERTIES

**Definition 2.1.** ([1]) A topological property  $P$  is called properly (respectively, closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property, if the following statement holds: if every proper (respectively, closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) subspace has the property  $P$ , then the whole space has the property  $P$ .

**Definition 2.2.** ([10]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise  $T_0$  if for any pair of distinct points  $x$  and  $y$  in  $X$ , there exist  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  such that  $x \in U, y \notin U$  or  $x \notin V, y \in V$ .

**Definition 2.3.** ([10]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise  $T_1$  if for any pair of distinct points  $x$  and  $y$  in  $X$ , there exist  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  such that  $x \in U, y \notin U$  and  $x \notin V, y \in V$ .

**Definition 2.4.** ([5]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise  $T_2$  if for any pair of distinct points  $x$  and  $y$  in  $X$ , there exist  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

**Definition 2.5.** ([5]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be  $\tau_1$  regular with respect to  $\tau_2$  if for any  $x \in X$  and  $\tau_1$ -closed set  $F$  in  $X$  not containing  $x$ , there exist  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  such that  $x \in U, F \subseteq V$  and  $U \cap V = \emptyset$ .

**Definition 2.6.** ([5]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise regular if, and only if, it is  $\tau_1$  regular with respect to  $\tau_2$  and  $\tau_2$  regular with respect to  $\tau_1$ .

**Definition 2.7.** ([5]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise normal if whenever  $A$  is  $\tau_1$ -closed set and  $B$  is  $\tau_2$ -closed set in  $X$  such that  $A \cap B = \emptyset$ , there exist  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  such that  $A \subseteq V, B \subseteq U$  and  $U \cap V = \emptyset$ .

**Definition 2.8.** ([9]) Let  $(X, \tau_1, \tau_2)$  be a bitopological space then:

- i)  $G$  is called pairwise open if, and only if,  $G$  is  $\tau_1$ -open and  $\tau_2$ -open in  $X$ .
- ii)  $F$  is called pairwise closed if, and only if,  $F$  is  $\tau_1$ -closed and  $\tau_2$ -closed in  $X$ .

**Theorem 2.1.** Let  $(X, \tau_1, \tau_2)$  be a bitopological space with more than two points. Then if every proper subspace of  $X$  is pairwise  $T_0$ , then  $X$  is pairwise  $T_0$ .

*Proof.* Let  $x \neq y$  in  $X$  and let  $z \in X \setminus \{x, y\}$ . Let  $A = X \setminus \{z\}$ . Since  $(A, \tau_{1A}, \tau_{2A})$  is pairwise  $T_0$ , then there exist  $\tau_{1A}$ -open set  $U_1$  and  $\tau_{2A}$ -open set  $V_1$  such that  $x \in U_1, y \notin U_1$  or  $x \notin V_1, y \in V_1$ , and hence there are  $\tau_1$ -open set  $U$  and

$\tau_2$ –open set  $V$  in  $X$  such that  $U_1 = U \cap A$  and  $V_1 = A \cap V$ ,  $x \in U$ ,  $y \notin U$  or  $x \notin V$ ,  $y \in V$ .  $\square$

**Theorem 2.2.** *Let  $(X, \tau_1, \tau_2)$  be a bitopological space with more than two points. Then if every proper subspace of  $X$  is pairwise  $T_1$ , then  $X$  is pairwise  $T_1$ .*

*Proof.* Let  $x \neq y$  in  $X$  and let  $z \in X \setminus \{x, y\}$ . Let  $A = X \setminus \{z\}$ . Since  $(A, \tau_{1A}, \tau_{2A})$  is pairwise  $T_1$ , then there exist  $\tau_{1A}$ –open set  $U_1$  and  $\tau_{2A}$ –open set  $V_1$  such that  $x \in U_1$ ,  $y \notin U_1$  and  $x \notin V_1$ ,  $y \in V_1$ , and hence there are  $\tau_1$ –open set  $U$  and  $\tau_2$ –open set  $V$  in  $X$  such that  $U_1 = U \cap A$  and  $V_1 = A \cap V$ ,  $x \in U$ ,  $y \notin U$  and  $x \notin V$ ,  $y \in V$ .  $\square$

**Theorem 2.3.** *Let  $(X, \tau_1, \tau_2)$  be a bitopological space with more than two points. Then if every proper subspace  $X$  is pairwise  $T_2$ , then  $X$  is pairwise  $T_2$ .*

*Proof.* Let  $x \neq y$  in  $X$  and let  $z \in X \setminus \{x, y\}$ . Let  $A = X \setminus \{z\}$ . Since  $(A, \tau_{1A}, \tau_{2A})$  is pairwise  $T_2$ , then there exist  $\tau_{1A}$ –open set  $U_1$  and  $\tau_{2A}$ –open set  $V_1$  such that  $x \in U_1$ ,  $y \notin U_1$  and  $x \notin V_1$ ,  $y \in V_1$  and  $U_1 \cap V_1 = \emptyset$ . Thus there are  $\tau_1$ –open set  $U$  and  $\tau_2$ –open set  $V$  in  $X$  such that  $U_1 = U \cap A$  and  $V_1 = A \cap V$ ,  $x \in U$ ,  $y \notin U$  and  $x \notin V$ ,  $y \in V$  and  $U \cap V = \emptyset$ .  $\square$

**Theorem 2.4.** *Let  $(X, \tau_1, \tau_2)$  be a bitopological space with more than two points. Then if every proper subspace  $X$  is pairwise regular, then  $X$  is pairwise regular.*

*Proof.* Let  $x \in X$  and  $F$  is  $\tau_1$ –closed set in  $X$  not containing  $x$ , let  $y \in X \setminus (\{x\} \cup F)$ . Let  $A = X \setminus \{y\}$ . Since  $(A, \tau_{1A}, \tau_{2A})$  is pairwise regular, then there exist  $\tau_{1A}$ –open set  $U_1$  and  $\tau_{2A}$ –open set  $V_1$  such that  $x \in U_1$ ,  $F \subseteq V_1$  and  $U_1 \cap V_1 = \emptyset$ , and hence there are  $\tau_1$ –open set  $U$  and  $\tau_2$ –open set  $V$  in  $X$  such that  $U_1 = U \cap A$  and  $V_1 = A \cap V$  such that  $x \in U$ ,  $F \subseteq V$  and  $U \cap V = \emptyset$ . Hence  $X$  is  $\tau_1$  regular with respect to  $\tau_2$ . By a similar method we can show that  $X$  is  $\tau_2$  regular with respect to  $\tau_1$ . Hence  $(X, \tau_1, \tau_2)$  is pairwise regular.  $\square$

**Corollary 2.1.** *Pairwise  $T_3$  is properly hereditary property.*

**Theorem 2.5.** *Let  $(X, \tau_1, \tau_2)$  be a bitopological space with more than two points. Then if every proper subspace  $X$  is pairwise normal, then  $X$  is pairwise normal.*

*Proof.* Let  $A$  be a  $\tau_1$ –closed set and  $B$  be a  $\tau_2$ –closed set in  $X$  such that  $A \cap B = \emptyset$ , let  $x \in X \setminus \{A \cup B\}$ . Let  $C = X \setminus \{x\}$ . Since  $(C, \tau_{1C}, \tau_{2C})$  is pairwise normal, then there exist  $\tau_{1C}$ –open set  $U_1$  and  $\tau_{2C}$ –open set  $V_1$  such that  $A \subseteq V_1$ ,  $B \subseteq U_1$

and  $U_1 \cap V_1 = \emptyset$ , and hence there are  $\tau_1$ -open set  $U$  and  $\tau_2$ -open set  $V$  in  $X$  such that  $U_1 = U \cap C$  and  $V_1 = C \cap V$ , such that  $A \subseteq V$ ,  $B \subseteq U$  and  $U \cap V = \emptyset$ .  $\square$

**Corollary 2.2.** *Pairwise  $T_4$  is properly hereditary property.*

**Definition 2.9.** ([4]) *A function  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1, \tau_2)$  is pairwise continuous (pairwise open, pairwise closed, pairwise homomorphism, respectively) if, and only if,  $f : (X, \tau_1) \rightarrow (Y, \tau_1)$  and  $f : (X, \tau_2) \rightarrow (Y, \tau_2)$  are continuous (open, closed, homomorphism, respectively).*

**Definition 2.10.** *A pairwise Hausdorff bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise minimal Hausdorff if, and only if, every one-to-one pairwise continuous function of  $X$  to a pairwise Hausdorff space  $Y$  is a pairwise homomorphism.*

**Theorem 2.6.** *Pairwise minimal Hausdorff is properly hereditary property.*

*Proof.* Suppose that every proper subspace of  $(X, \tau_1, \tau_2)$  is pairwise minimal Hausdorff. Since every proper subspace of  $(X, \tau_1, \tau_2)$  is pairwise Hausdorff, then  $(X, \tau_1, \tau_2)$  is pairwise Hausdorff. Let  $f$  be a one-to-one pairwise continuous function of  $(X, \tau_1, \tau_2)$  to a pairwise Hausdorff space  $(Y, \tau_1, \tau_2)$  and let  $x \in X$ . Let  $A = X \setminus \{x\}$ , then  $h : (A, \tau_1) \rightarrow (Y - f(\{x\}), \tau_1)$  and  $h : (A, \tau_2) \rightarrow (Y - f(\{x\}), \tau_2)$  is one-to-one pairwise continuous function where  $h = f$  on  $A$  and hence  $h$  is pairwise homomorphism, since  $f(X) = f(X - \{x\}) \cup f(\{x\}) = f(A) \cup f(\{x\}) = h(A) \cup f(\{x\}) = Y \setminus f(\{x\}) \cup f(\{x\}) = Y$ , so  $f$  is onto.

Let  $U$  pairwise open set in  $X$ , if  $X \neq U$ , we have two cases:

(i) if  $x \notin U$ , then  $f(U) = h(U)$ , and since  $h$  is pairwise homomorphism we have  $h(U)$  is pairwise open in  $Y - f(\{x\})$  so it is pairwise open in  $Y$ .

(ii) if  $x \in U$ , let  $y \in X \setminus U$ . Let  $B = X \setminus \{y\}$ . So  $g : B \rightarrow Y \setminus f(\{y\})$  is one-to-one pairwise continuous map where  $g = f$  on  $B$  and hence  $g$  is pairwise homomorphism that implies  $f(U) = g(U)$  is pairwise open in  $Y \setminus g(\{y\})$  so it is pairwise open in  $Y$ .

From (i) and (ii) we have  $f$  is pairwise open, so  $f$  is pairwise homomorphism.

If  $U = X$ , since  $f$  is onto, then  $f(U) = Y$  so it is pairwise open, so  $f$  is pairwise homomorphism.  $\square$

**Definition 2.11.** ([3]) *A cover  $\underline{U} = \{U_\alpha : \alpha \in \Delta\}$  is said to be pairwise open cover of  $X$  if, and only if,  $\underline{U} \subseteq \tau_1 \cup \tau_2$  and for each  $i \in \{1, 2\}$ ,  $\underline{U} \cap \tau_i$  contains at least a non empty set.*

**Definition 2.12.** ([3]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise compact if for every pairwise open cover of the space  $X$  has a finite subcover.

**Definition 2.13.** ([3]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise Lindelöf if for every pairwise open cover of the space  $X$  has a countable subcover.

**Definition 2.14.** ([3]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise countably compact if for every pairwise countable open cover of the space  $X$  has a finite subcover.

**Theorem 2.7.** The pairwise compactness is a closed hereditary property (i.e if every pairwise closed subset  $A$  of a pairwise compact space  $(X, \tau_1, \tau_2)$  is pairwise compact then  $(X, \tau_1, \tau_2)$  is pairwise compact).

*Proof.* Let  $\underline{U} = \{U_\alpha : \alpha \in \Delta\}$  be a pairwise open cover of a bitopological space  $(X, \tau_1, \tau_2)$ . Pick  $U_{\alpha_0} \in \underline{U}$  with  $U_{\alpha_0} \neq \emptyset$ . Let  $A = X \setminus U_{\alpha_0}$ , then  $A$  is a pairwise closed subspace of  $(X, \tau_1, \tau_2)$ . Therefore  $(A, \tau_{1A}, \tau_{2A})$  is pairwise compact. Then  $\{U_\alpha : \alpha \in \Delta\} \setminus \{U_{\alpha_0}\}$  is a pairwise open cover of  $A$ , hence it has finite subcover  $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ . Therefore  $\{U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$  is a finite subcover of  $X$ , so  $X$  is pairwise compact.  $\square$

**Theorem 2.8.** Pairwise countably compactness and pairwise Lindelöfness space are closed hereditary property.

*Proof.* The proof follows by a similar method that used in Theorem 2.7.  $\square$

**Definition 2.15.** ([8]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be  $\tau_1$  locally compact with respect to  $\tau_2$  if for each point  $x \in X$ , there is  $\tau_1$ -open neighbourhood of  $x$  whose  $\tau_2$ -closure is compact.

**Definition 2.16.** ([8]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise locally compact if, and only if, it is  $\tau_1$  locally compact with respect to  $\tau_2$  and  $\tau_2$  locally compact with respect to  $\tau_1$ .

**Theorem 2.9.** Pairwise  $T_1$ , pairwise locally compact space is properly hereditary property.

*Proof.* Let  $(X, \tau_1, \tau_2)$  be a bitopological space, since pairwise  $T_1$  is properly hereditary property, then  $(X, \tau_1, \tau_2)$  is pairwise  $T_1$ . Let  $x \in X$  and let  $y \in X \setminus \{x\}$ . Let  $A = X \setminus \{y\}$ , since  $(A, \tau_{1A}, \tau_{2A})$  is pairwise locally compact, there is  $\tau_{1A}$ -open

neighbourhood  $U$  of  $x$  whose  $\tau_{2A}$ -closure is compact, and since  $(X, \tau_1, \tau_2)$  is pairwise  $T_1$ , then  $U$  is  $\tau_1$ -open neighbourhood of  $x$  whose  $\tau_2$ -closure is compact then  $(X, \tau_1, \tau_2)$  is  $\tau_1$  locally compact with respect to  $\tau_2$ .

Using the same technique we can show that  $(X, \tau_1, \tau_2)$  is  $\tau_2$  locally compact with respect to  $\tau_1$ .  $\square$

**Definition 2.17.** ([2]) If  $\underline{U}$  is a pairwise open cover of  $(X, \tau_1, \tau_2)$ , then the pairwise open cover  $\underline{E}$  of  $(X, \tau_1, \tau_2)$  is said to be a parallel refinement of  $\underline{U}$  if  $\underline{E} \cap \tau_i$  refines  $\underline{U} \cap \tau_i$ ,  $i = 1, 2$ .

**Definition 2.18.** ([2]) A refinement  $\underline{E}$  of a pairwise open cover  $\underline{U}$  of  $(X, \tau_1, \tau_2)$  is said to be pairwise locally finite if for each point  $x$  of  $X$ , there is a  $\tau_1$ -open or  $\tau_2$ -open set  $U$  containing  $x$  and intersect finitely many member of  $\underline{E}$ .

**Definition 2.19.** ([2]) A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise paracompact if every pairwise open cover of  $X$  has pairwise open locally finite parallel refinement.

**Remark 2.1.** Let  $X$  be a pairwise paracompact, pairwise Housdorff. Then every pairwise closed subset  $A$  of  $X$  has the property  $P$  which says: every pairwise open cover of  $A$  (in  $X$ ) has a pairwise open parallel refinement which is locally finite in  $X$ .

The proof follows easily.

**Remark 2.2.** Every subset  $A$  of a bitopological space which has the property  $P$  is pairwise paracompact.

**Theorem 2.10.** If  $X$  is pairwise Housdorff space such that, every pairwise closed subset  $A$  of  $X$  has the property  $P$ , then  $X$  is pairwise paracompact.

*Proof.* Let  $\underline{U} = \{U_\alpha : \alpha \in \Delta\}$  be pairwise open cover of a bitopological space  $(X, \tau_1, \tau_2)$ . Let  $U_{\alpha_0}$  be fixed element of  $\underline{U}$ , then  $X \setminus U_{\alpha_0}$  is pairwise closed hence  $X \setminus U_{\alpha_0}$  has the property  $P$ . Since  $\underline{U}$  is a pairwise open cover of  $X \setminus U_{\alpha_0}$ , it has a pairwise open parallel locally finite refinement, say  $\underline{E}$ . Then  $\underline{E} \cup \{U_{\alpha_0}\}$  is a pairwise open parallel locally finite refinement of  $\underline{U}$ . Hence  $X$  is pairwise paracompact.  $\square$

**Definition 2.20.** A bitopological space  $(X, \tau_1, \tau_2)$  is said to be pairwise  $\sigma$ -compact if, and only if,  $X$  is a countable union of pairwise compact subset.

**Theorem 2.11.** *A pairwise  $\sigma$ -compact is properly hereditary property.*

*Proof.* Let  $x \in X$ . Let  $Y = X \setminus \{x\}$ . Since every proper subset of  $X$  is pairwise  $\sigma$ -compact then there exist a pairwise compact subset  $A_1, A_2, A_3, \dots$  in  $Y$ , and hence in  $X$  such that  $\cup_{i=1} A_i = Y$ . So  $\{x\}, A_1, A_2, A_3, \dots$  are pairwise compact subset in  $X$  such that  $\cup_{i=1} A_i \cup \{x\} = X$ .  $\square$

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