

COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF GENERALIZED FRESNEL COSINE INTEGRALS AND APPLICATIONS

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ABSTRACT. The generalized Fresnel cosine integral $C_k(x)$ and its associated functions $C_{k+}(x)$ and $C_{k-}(x)$ are defined as locally summable functions on the real line. The generalized Fresnel cosine integrals have huge applications in physics, specially in optics and electromagnetics. In many diffraction problems the generalized Fresnel integrals plays an important role. In this paper are calculated the commutative neutrix convolutions of the generalized Fresnel cosine integral and its associated functions with $x^r, r = 0, 1, 2, \dots$.

1. INTRODUCTION AND PRELIMINARIES

The generalized Fresnel integrals are transcendent functions, which are named by Augustin-Jean Fresnel who used its in the optics. The generalized Fresnel sine and Fresnel cosine integrals have huge applications in physics, especially in optics and electromagnetics. In many diffraction problems the generalized Fresnel integrals play an important role. Fresnel integrals were first used for calculation of the intensity of electromagnetic field when the opaque object is shrouded in light, [1]. Later, they were used in construction, engineering, road construction and railways, [2]. In [3], the generalized Fresnel integral seems as a canonical function uniform ray field representation of several high-frequency diffraction

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mechanisms. The generalized Fresnel integral was first introduced in [4] and in [5] its properties were analyzed. The generalized Fresnel integrals are used in vertex problem and many other diffraction problems. They are used also in the simulation for increasing of the electromagnetic waves, [6]. Its applications becomes more practical if the generalized Fresnel integrals are presented in closed form expressions. Direct numerical calculation of this integral is complicated because of the rapid oscillations of the integrand that occur for certain combinations of the two real arguments, [3]. In [7], the authors introduced two approximations for analytical evaluation of the Fresnel integrals without using numerical algorithms.

But, not only the calculation of the generalized Fresnel integrals is complicated. The calculation of many products and convolution products which involving the generalized Fresnel integrals is also complicated in the functional analysis. Because of that, in the functional analysis were introduced new approaches like calculations of convolutions of distributions and products of distributions in neutrix sense introduced by van der Corput in [8], [9], or calculation of products of distribution in Colombeau algebra introduced by Colombeau in [10, 11].

The generalized Fresnel cosine integral $C_k(x)$ is defined by:

$$C_k(x) = \int_0^x \cos u^k du, \quad k = 1, 2, \dots,$$

in [12] and its associated functions $C_{k+}(x)$ and $C_{k-}(x)$ are defined by:

$$C_{k+}(x) = H(x)C_k(x), \quad C_{k-}(x) = H(-x)C_k(x),$$

where H is Heaviside's function.

We define the function

$$(1.1) \quad I_{r,k}(x) = \int_0^x u^r \cos u^k du$$

for $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Also, we define functions $\cos_+ x^k$ and $\cos_- x^k$ by

$$(1.2) \quad \cos_+ x^k = H(x) \cos x^k, \quad \cos_- x^k = H(-x) \cos x^k.$$

In mathematical analysis some operations like product of distributions or convolution product of distributions cannot be defined for arbitrary distributions. The convolution product in classical sense is defined by the following definition:

Definition 1.1. *Let f and g be two functions. The convolution product $f * g$ is defined by the equation:*

$$(1.3) \quad (f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt$$

in all points x for which the integral exists.

From the definition 1.1, it follows that in classical sense if the convolution product $f * g$ exists, then $g * f$ exists and

$$(1.4) \quad f * g = g * f,$$

and moreover if $(f * g)'$ exists and $f' * g$ (or $f * g'$) exists and

$$(1.5) \quad (f * g)' = f' * g = f * g'.$$

The definition 1.1 cannot be used for arbitrary distributions, because some operations as product, convolution product cannot be defined in general. The following definition which is given in [14] extends the definition 1.1 for distributions f and g in the space \mathcal{D}' .

Definition 1.2. *Let f and g be two distributions in \mathcal{D}' . Then the convolution product $f * g$ is defined by the equation*

$$(1.6) \quad \langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitray test function $\varphi \in \mathcal{D}$, when f and g satisfy the following conditions:

- (1) *f and g have bounded support;*
- (2) *the supports of f and g are bounded on the same side.*

In [15] are calculated the convolution products of the generalized Fresnel cosine integral $C_k(x)$ with the distribution x^r .

The definition 1.2 was extended by Fisher in [9] in order to define convolution product to a larger class of distributions. In that way, Fisher has defined the non-commutative neutrix convolution product, in [9]. For that extension, he defined the function τ in \mathcal{D}' which satisfies the following properties:

- (i) $\tau(x) = \tau(-x)$;
- (ii) $0 \leq \tau(x) \leq 1$;
- (iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$;
- (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_ν is defined for $\nu > 0$ by:

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu \end{cases}.$$

The following definition of the non-commutative neutrix convolution was given in [9].

Definition 1.3. Let f and g be distributions in \mathfrak{D}' and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the non-commutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided the limit h exists in the sense that:

$$N - \lim_{\nu \rightarrow \infty} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all $\varphi \in \mathfrak{D}$, where N is neutrix, see [8], having domain N' the positive real numbers and negligible functions finite linear sums of the functions

$$\nu^\lambda \ln^{r-1} \nu, \ln^r \nu, \nu^r \sin \nu^k, \nu^r \cos \nu^k : \lambda > 0, r = 1, 2, \dots, k = 1, 2, \dots,$$

and all functions which converge to zero in normal sense when ν tends to infinity.

Any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution product. Actually, the neutrix convolution is a generalization of the convolution, which is shown in [9], i.e.

$$f \circledast g = f * g.$$

The next definition for commutative neutrix product is given in [16]:

Definition 1.4. Let f and g be distributions in \mathfrak{D}' and let $f_\nu = f\tau_\nu$ and $g_\nu = g\tau_\nu$ for $\nu > 0$. The commutative neutrix convolution $f \boxtimes g$ is defined as the neutrix limit of the sequence $\{f_\nu * g_\nu\}$, provided the limit h exists in the sense that:

$$N - \lim_{\nu \rightarrow \infty} \langle f_\nu * g_\nu, \varphi \rangle = \langle h, \varphi \rangle$$

for all $\varphi \in \mathfrak{D}$, where N is neutrix, see [8], having domain N' the positive real numbers and negligible functions finite linear sums of the functions

$$v^\lambda \ln^{r-1} v, \ln^r v, v^r \sin v^k, v^r \cos v^k : \lambda > 0, r = 1, 2, \dots, k = 1, 2, \dots,$$

and all functions which converge to zero in normal sense when v tends to infinity.

In the next section will be proved theorems and corollaries for generalized Fresnel integrals, which are generalization of the results proved in [17]. In order to prove our results we will lemma 8, which is proved in [13].

2. MAIN RESULTS

We prove the following theorem:

Theorem 2.1. *The commutative neutrix convolution $(\cos_+ x^k) \boxtimes x^r$ exists and*

$$(2.1) \quad (\cos_+ x^k) \boxtimes x^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i,k} x^i,$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$.

Proof. Because we need to calculate commutative neutrix convolution product, according to the definition 1.4 we define the sequence of regular distributions $(\cos_+ x^k)_v = (\cos_+ x^k) \tau_v(x)$ and $(x^r)_v = x^r \tau_v(x)$. Then the convolution $(\cos_+ x^k)_v * (x^r)_v$ exists,

$$(2.2) \quad \begin{aligned} (\cos_+ x^k)_v * (x^r)_v &= \int_0^v \cos t^k (x-t)^r \tau_v(x-t) dt \\ &+ \int_v^{v+v^{-v}} \cos t^k (x-t)^r \tau_v(t) \tau_v(x-t) dt = I_1 + I_2. \end{aligned}$$

If $0 \leq |x| \leq v$, then for the first integral we have:

$$\begin{aligned} \int_0^v \cos t^k (x-t)^r \tau_v(x-t) dt &= \int_0^v \cos t^k \sum_{i=0}^r \binom{r}{i} x^i (-t)^{r-i} \tau_v(x-t) dt \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i,k}(v) x^i. \end{aligned}$$

So for the neutrix limit of this integral, according to lemma 8 in [13], we obtain:

$$(2.3) \quad N - \lim_{\nu \rightarrow \infty} \int_0^{\nu} \cos t^k (x-t)^r \tau_{\nu}(x-t) dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i,k} x^i.$$

For the second integral from the defined function $\tau_{\nu}(t)$ and because $|\cos t^k| \leq 1$ it follows:

$$\left| \int_{\nu}^{\nu+\nu^{-\nu}} \cos t^k (x-t)^r \tau_{\nu}(t) \tau_{\nu}(x-t) dt \right| \leq \int_{\nu}^{\nu+\nu^{-\nu}} (x-t)^r dt \leq (\nu + \nu^{-\nu}) \nu^{-\nu},$$

so for fixed x it follows:

$$(2.4) \quad \lim_{\nu \rightarrow \infty} \int_{\nu}^{\nu+\nu^{-\nu}} \cos t^k (x-t)^r \tau_{\nu}(t) \tau_{\nu}(x-t) dt = 0.$$

So the equation (2.1) follows from the equations (2.2), (2.10) and (2.11). \square

Corollary 2.1. *The commutative neutrix convolution $(\cos_- x^k) \boxtimes x^r$ exists and*

$$(2.5) \quad (\cos_- x^k) \boxtimes x^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i+1} I_{r-i,k} x^i,$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$

Proof. If in the equation (2.1) we replace x by $-x$, then we obtain the equation (2.5). Here, also is noticed that

$$(2.6) \quad N - \lim_{\nu \rightarrow \infty} I_{r,k}(-\nu) = (-1)^{r-1} N - \lim_{\nu \rightarrow \infty} I_{r,k}(\nu) = (-1)^{r-1} I_{r,k}.$$

\square

Corollary 2.2. *The commutative neutrix convolution $(\cos x^k) \boxtimes x^r$ exists and*

$$(2.7) \quad (\cos x^k) \boxtimes x^r = 0,$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$

Proof. The equation (2.7) follows from the equations (2.5) and (2.1) and from the equation $\cos x^k = \cos_+ x^k + \cos_- x^k$. \square

In the next theorem and two corollaries it will be calculated the commutative neutrix convolution product of the generalized Fresnel cosine integral with the distribution $x^r, r = 0, 1, 2, \dots$

Theorem 2.2. *The commutative neutrix convolution $C_{k+}(x) \boxtimes x^r$ exists and*

$$(2.8) \quad C_{k+}(x) \boxtimes x^r = \frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1,k} x^i$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$

Proof. In order to calculate commutative neutrix convolution product of the distributions we define the sequence $(C_{k+}(x))_\nu = C_{k+}(x)\tau_\nu(x)$ and $(x^r)_\nu = x^r\tau_\nu(x)$. Then the convolution $(C_{k+}(x))_\nu * (x^r)_\nu$ exists.

$$(2.9) \quad \begin{aligned} (C_{k+}(x))_\nu * (x^r)_\nu &= \int_0^\nu C_k(t)(x-t)^r \tau_\nu(x-t) dt \\ &+ \int_\nu^{\nu+\nu^{-\nu}} C_k(t)t(x-t)^r \tau_\nu(t) \tau_\nu(x-t) dt = I_1 + I_2. \end{aligned}$$

If $0 \leq |x| \leq \nu$, then for the first integral we have:

$$\int_0^\nu C_k(t)(x-t)^r \tau_\nu(x-t) dt = \int_0^\nu (x-t)^r \int_0^t \cos u^k du dt.$$

So for the neutrix limit of this integral, according to lemma 8 in [13], we obtain:

$$(2.10) \quad N - \lim_{\nu \rightarrow \infty} \int_0^\nu \cos t^k (x-t)^r \tau_\nu(x-t) dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i,k} x^i.$$

For the second integral from the defined function $\tau_\nu(t)$ and because $|\cos t^k| \leq 1$ it follows:

$$\left| \int_\nu^{\nu+\nu^{-\nu}} \cos t^k (x-t)^r \tau_\nu(t) \tau_\nu(x-t) dt \right| \leq \int_\nu^{\nu+\nu^{-\nu}} (x-t)^r dt \leq (\nu + \nu^{-\nu}) \nu^{-\nu},$$

so for fixed x it follows:

$$(2.11) \quad \lim_{\nu \rightarrow \infty} \int_\nu^{\nu+\nu^{-\nu}} \cos t^k (x-t)^r \tau_\nu(t) \tau_\nu(x-t) dt = 0.$$

So the equation (2.1) follows from the equations (2.2), (2.10) and (2.11). \square

Corollary 2.3. *The commutative neutrix convolution $C_{k-}(x) \boxtimes x^r$ exists and*

$$(2.12) \quad C_{k-}(x) \boxtimes x^r = \frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i} I_{r-i+1,k} x^i$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$

Proof. The equation (2.12) follows from the equation (2.8) by replacing x by $-x$. \square

Corollary 2.4. *The commutative neutrix convolution $C_k(x) \boxtimes x^r$ exists and*

$$(2.13) \quad C_k(x) \boxtimes x^r = 0$$

for $r = 0, 1, 2, \dots$ and $k = 1, 2, \dots$

Proof. The equation (2.13) follows from the equations (2.8) and (2.12) and noting that $C_k(x) = C_{k+} + C_{k-}$. \square

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