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CONVERGENCE OF THE SIMPLE EXACT BARRIER-PENALTY FUNCTION FOR NONLIEAR MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. In this paper, an extension of the new barrier penalty technical is proposed. Initially, design for nonlinear single-objective optimization with inequality constraints, we have transformed it for solving nonlinear multiobjective optimization problems with inequality constraints. First, we have provided the theoretical foundations for this extension. Secondly, we have stated convergence results of our new method to obtain Pareto optimal solutions. This work shows that the new penalty technique converges well for the determination of Pareto optimal solutions of a multiobjective optimization problems with inequality constraints.

1. INTRODUCTION

Multi-objective optimization consists in simultaneously optimizing several objective functions with or without constraints. Mathematical modeling for everyday life problems is much more realistic when it takes into consideration only one objective. The resulting mathematical programs do not have a single solution, which is said to be bad in the antique concepts of the mathematics. In addition, the objectives are generally conflicting and contribute to making the search for solutions very difficult. The solution to a multi-objective optimization problem is to identify a set of solutions that best optimize each of the objectives.

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These solutions are called "Pareto optimal solutions" to characterize the fact that they are solutions that do not dominate each other.

In the literature, there are many methods dedicated to the search for these types of solutions. But, there is not universal method that is effectively applied to all family of the multiobjective problems. Most of these methods transform constrained optimization problems into unconstrained optimization problems to facilitate resolution. This transformation is achieved through the use of penalization functions (see in [3, 7, 8, 12, 14, 16, 17]). Penalization techniques are used in both single objective and multi-objective cases, regardless of the nature of the variables in the problem. Among the multiples techniques of penalizing of literature, the one defined in [3], called logarithmic barrier-penalty function, which caught our attention. So far, it has been used only for the single-objective case. Indeed, for a problem of mono-objective optimization form:

$$\begin{aligned} &\min\ \varphi(x);\\ &s.t:\begin{cases} h_j(x)\le 0, j=1,\cdots,l;\\ x\in\mathbb{R}^n; \end{cases} \end{aligned} \tag{1.1}$$

where $X=\{x\in\mathbb{R}^n|h_j(x)\le 0, j=1,\cdots,l\}$ is the admissible set of problem (1.1). The penalty function proposed by Bingzhuang Liu [3] to transform this problem in unconstrained problem is defined as follows:

$$P_\sigma(x,\tau)=\begin{cases} \varphi(x), & \text{if } \tau=0, x\in X;\\ \varphi(x)-\tau\sum_{j=1}^{l}\ln(\tau-h_j(x))+\tau\sigma, & \text{if } \tau>0, x\in X_\tau;\\ +\infty, & \text{somewhere else}; \end{cases} \tag{1.2}$$

where $X_\tau=\{x\in\mathbb{R}^n|h_j(x)<\tau, j=1,\cdots,l\}$ with the operations on X_τ can be extended to the admissible domain X of the initial problem (1.1) (see in [3]) and σ a penalty parameter. Furthermore, it does not take into account the differentiation of objective or constrained functions. In search of better methods of solving multi-objective optimization problems, several concepts related to single-objective have been extended. Examples of works of this type can be found in [4]; [13]; [6] and [5], which proposed an extension of the exact penalty functions; [1]; [**?**] and [9] which presented an extension of the exponential penalty functions; [10]which also proposed an extension of exponential penalty functions for the particular case of non-differential functions and [15] which proposed an extension of the logarithmic penalty function for the fractional objective functions.

In this work, we have proposed an extension of the logarithmic barrier-penalty function for solving the multiobjective optimization problems. Considering the advantages offered by this penalty function, our approach is able to find the Pareto optimal solutions to many kinds of multiobjective optimization problems. To prove the effectiveness of our approach, we have formulated and demonstrated the theoretical results of convergence towards optimal Pareto solutions.

This article is organized as follows: in Section 2, we will present the basic concepts; in Section 3, we will present the results of the convergence of our approach and in Section 4, make a conclusion.

2. Multiobjective Optimization Concepts

Let us consider the following multiobjective optimization problem with inequality constraints:

$$\begin{array}{ll} \min & f_1(x); \\ \min & f_2(x); \\ & \vdots \\ \min & f_p(x); \\ s.t: & \left\{ \begin{array}{c} g_1(x) \leq 0; \\ g_2(x) \leq 0; \\ \vdots \\ g_m(x) \leq 0; \\ x \in \mathbb{R}^n; \end{array} \right. \end{array} \tag{2.1}$$

with $f_1(x), f_2(x), \cdots, f_p(x)$ the objectives functions and $g_1(x), g_2(x), \cdots, g_m(x)$ constrained functions. To simplify the notations, let us set $f(x) = (f_1(x), f_2(x), \cdots, f_p(x))^T$, the objective function vector and $g(x) = (g_1(x), g_2(x), \cdots, g_m(x))^T$ the constraint function vector. So, we can reword the problem (2.1) by:

$$\begin{array}{l} \min\ f(x); \\ s.t: \left\{ \begin{array}{l} g(x) \leq 0 \\ x \in \mathbb{R}^n. \end{array} \right. \end{array} \tag{2.2}$$

A very important element in multiobjective optimization is the admissible domain that we will define by $\chi = \{x \in \mathbb{R}^n,\ g(x) \leq 0\}$. The following definition allows us to best understand its concept.

Definition 2.1.

- *A point $x^* \in \chi$ of the problem (2.1) is weakly Pareto optimal if and only if there is no point $x \in \chi$ such that:$f_j(x) < f_j(x^*), \forall j = 1, \cdots, p$.*
- *A point $x^* \in \chi$ is Pareto optimal of the problem (2.1) if there is no solution $x \in \chi$ such that $f_j(x) \leq f_j(x^*)$, $\forall j = 1, \cdots, p$ and for at least one $k \in \{1, \cdots, p\}, j \neq k$, we have $f_k(x) < f_k(x^*)$.*

Throughout this paper, we will denote by $\mathcal{W}^*$ (resp. $\mathcal{P}^*$) the set of weakly Pareto optimal solutions (resp. the set of Pareto optimal solutions) of the problem (2.1).

3. Our Approach

3.1. Extended Exact Penalty Barrier Function.

Similar to the barrier penalty approach in the single-objective case, let us construct the unconstrained multiobjective optimization problem as follows:

$$\text{(3.1)} \qquad \begin{cases} \min & f_1(x) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x)) + \sigma_n \tau_n; \\ \min & f_2(x) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x)) + \sigma_n \tau_n; \\ \vdots & \vdots \\ \min & f_p(x) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x)) + \sigma_n \tau_n; \\ x \in \mathbb{R}^n; \end{cases}$$

with $\tau_n > 0$ a decreasing sequence such as $\lim_{n \to +\infty} \tau_n = 0$. Suppose here that $(\sigma_n)_n$ is an infinitely increasing and bounded sequence. For the problem (3.1), we will respectively designate by $\mathcal{W}_n^*$ the set of weakly Pareto optimal solutions and $\mathcal{P}_n^*$ the set of its Pareto optimal solutions.

Let $\mathcal{Q}_n \subset \mathbb{R}^k, n \in K = \{1, 2, \cdots\}$, denote by

$$\overline{\lim_{n\to+\infty}} \mathcal{Q}_n = \bigcap_n \Big(\bigcup_{k\geq n} \mathcal{Q}_k \Big) = \{x \in \mathbb{R}^k : \text{ for infinitely many } n \text{ in } K\}$$

$$\underline{\lim_{n\to+\infty}} \mathcal{Q}_n = \bigcup_n \Big(\bigcap_{k\geq n} \mathcal{Q}_k \Big) = \{x \in \mathbb{R}^k : \text{for all but finitely many } n \text{ in } K\}.$$

We have $\underline{\lim_{n\to+\infty}} \mathcal{Q}_n \subseteq \overline{\lim_{n\to+\infty}} \mathcal{Q}_n$. If $\overline{\lim_{n\to+\infty}} \mathcal{Q}_n \subseteq \underline{\lim_{n\to+\infty}} \mathcal{Q}_n$ then $\lim_{n\to+\infty} \mathcal{Q}_n = \underline{\lim_{n\to+\infty}} \mathcal{Q}_n = \overline{\lim_{n\to+\infty}} \mathcal{Q}_n$.

Let $\chi_\tau = \{x \in \mathbb{R}^n | g_j(x) < \tau, j = 1, \cdots, m\}$ the admissible domain of the problem (3.1) using the barrier penalty. We have the following lemmas:

Lemma 3.1.

(1) *If* $x \in \chi_\tau$, *then* $\lim_{n\to+\infty} (-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x)) + \sigma_n \tau_n) = 0$;

(2) *If* $x \notin \chi_\tau$, *then* $\lim_{n\to+\infty} (-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x)) + \sigma_n \tau_n) = +\infty$.

Lemma 3.2. *Let* $\tau_n > 0$ *and* $\lim_{n\to+\infty} \tau_n = 0$.

i. *If* $x^* \in \overline{\lim_{n\to+\infty}} W_n^*$ *then* $x^* \in \chi_\tau$;

ii. *If* $x^* \in \underline{\lim_{n\to+\infty}} \mathcal{W}_n^*$ *then* $x^* \in \chi_\tau$.

Proof.

i. Suppose that $x^* \in \overline{\lim_{n\to+\infty}} \mathcal{W}_n^*$, then there is a subsequence $\{n_k\}$ of K such that $x^* \in \mathcal{W}_{n_k}^*, k = 1, 2, \cdots$. Using the weakly Pareto optimality definition of the problem (3.1), there is no $\overline{x} \in \chi_\tau$ such that

$$f_i(\overline{x}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} < f_i(x^*) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}, k = 1, 2, \cdots, i = 1, \cdots, p. \tag{3.2}$$

Then, $x^* \notin \chi_\tau$ there is at least one index $j \in \{1, 2, \cdots, m\}$ such as $g_j(x^*) \geq \tau$. Through the second equality of Lemma 3.1, we have

$$\lim_{k\to+\infty} (-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}) = +\infty$$

and for any other point $\overline{x} \in \chi_\tau$, we have $\lim\limits_{k\to+\infty}(-\tau_{n_k}\sum\limits_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k}) = 0$.

So from a sufficiently large rank k_0 such as $k > k_0$, we have

$$\begin{aligned} &-\tau_{nk}\sum_{j=1}^{m}\ln(\tau_{nk} - g_j(\overline{x})) + \sigma_{n_k}\tau_{nk} \\ &< -\tau_{nk}\sum_{j=1}^{m}\ln(\tau_{nk} - g_j(x^*)) + \sigma_{n_k}\tau_{nk}. \end{aligned}$$

Therefore for, $k > k_0$ and for $i = 1, \cdots, p$, we have

$$\begin{aligned} &f_i(\overline{x}) - \tau_{nk}\sum_{j=1}^{m}\ln(\tau_{nk} - g_j(\overline{x})) + \sigma_{n_k}\tau_{nk} \\ &< f_i(x^*) - \tau_{nk}\sum_{j=1}^{m}\ln(\tau_{nk} - g_j(x^*)) + \sigma_{n_k}\tau_{nk} \end{aligned}$$

which contradicts the relation (3.2). Hence the first point of the Lemma 3.2.

ii. As $\underline{\lim}_{n\to+\infty}\mathcal{W}_n^* \subset \overline{\lim}_{n\to+\infty}\mathcal{W}_n^*$ then if $x^* \in \underline{\lim}_{n\to+\infty}\mathcal{W}_n^*$ we will have $x^* \in \overline{\lim}_{n\to+\infty}\mathcal{W}_n^*$. Therefore, according to the first point the Lemma 3.2, $x^* \in \chi_\tau$.

□

Lemma 3.3. *Let* $\tau_n > 0$ *and* $\lim\limits_{n\to+\infty}\tau_n = 0$.

(1) *If* $x^* \in \overline{\lim}_{n\to+\infty}\mathcal{P}_n^*$ *then* $x^* \in \chi_\tau$.
(2) *If* $x^* \in \underline{\lim}_{n\to+\infty}\mathcal{P}_n^*$ *then* $x^* \in \chi_\tau$.

Proof.

i. Since $x^* \in \overline{\lim}_{n\to+\infty}\mathcal{P}_n^*$ then there is a subsequence $\{n_k\}$ of K such as $x^* \in \mathcal{P}_{n_k}^*$. So there is no $\overline{x} \in \chi_\tau$ such that

$$f_i(\overline{x}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} \leq f_i(x^*) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}, k = 1, 2, \cdots, i = 1, \cdots, p \tag{3.3}$$

and for at least one $l \in \{1, \cdots, p\}$ we have: $f_l(\overline{x}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} \leq f_l(x^*) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}$. If $\overline{x} \in \chi_\tau$, then according to Lemma 3.1, $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k}) = 0.$

If $x^* \notin \chi_\tau$, then, according to Lemma 3.1, $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}) = +\infty.$

So from a certain rank k_0 such as $k > k_0$, we have $-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma\tau_{n_k} < -\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}$. Therefore,

$$f_i(\overline{x}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} \leq f_i(x^*) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*)) + \sigma_{n_k}\tau_{n_k}, \forall i = 1, \cdots, p. \tag{3.4}$$

We get a contradiction of (3.3). So, if $x^* \in \overline{\lim\limits_{n\longrightarrow+\infty}} \mathcal{P}_n^*$ then $x^* \in \chi_\tau$.

ii. Since $\underline{\lim\limits_{n\to+\infty}} \mathcal{P}_n^* \subset \overline{\lim\limits_{n\to+\infty}} \mathcal{P}_n^*$ so if $x^* \in \underline{\lim\limits_{n\to+\infty}} \mathcal{P}_n^*$ then $x^* \in \overline{\lim\limits_{n\to+\infty}} \mathcal{P}_n^*$. Therefore, according to the first point the Lemma 3.3, $x^* \in \chi_\tau$.

□

3.2. **Convergence Studies.**

This section is devoted to the study of the convergence of the penalization function in the multiobjective case.

3.2.1. *Towards weakly Pareto optimal solutions.*

Theorem 3.1. $\underline{\lim\limits_{n\to+\infty}} \mathcal{W}_n^* \setminus \mathcal{W}^* = \emptyset.$

Proof. Suppose $\underline{\lim\limits_{n\to+\infty}} \mathcal{W}_n^* \setminus \mathcal{W}^* \neq \emptyset$. Then, there is at least one $y \in \underline{\lim\limits_{n\to+\infty}} \mathcal{W}_n^* \setminus \mathcal{W}^*$, i.e. there exists a rank $n_0 > 0$ such as for $n \geq n_0$ we have $y \in \mathcal{W}_n^* \setminus \mathcal{W}^*$. It follows that $y \notin \mathcal{W}^*$. So:

- if $y \in \chi_\tau$, then there is a $\overline{y} \in \chi_\tau$ such that $f_i(\overline{y}) < f_i(y), \forall i = 1, \cdots, p$. Therefore, from a certain rank n_0, we have the inequality

$$f_i(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$< f_i(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n,$$

  $\forall i = 1, \cdots, p$, which contradicts the fact that $y \in \mathcal{W}_n^*$.

- if $y \notin \chi_\tau$, then, by Lemma 3.1, $\lim\limits_{n\to+\infty}(-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n) = +\infty$.

  Given a $\overline{y} \in \chi_\tau$, we have, by Lemma 3.1, $\lim\limits_{n\to+\infty}(-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n) = 0$. So from a certain rank $n_0 > 0$ such as $n \geq n_0$, we have:

$$f_i(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$< f_i(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n,$$

  $\forall i = 1, \cdots, p$ that is a contradiction relation to $y \in \mathcal{W}_n^*$.

Therefore, we can conclude that $\underline{\lim}_{n\to+\infty} \mathcal{W}_n^* \setminus \mathcal{W}^* \neq \emptyset$. That is absurd hence, the result. □

Theorem 3.2. $\overline{\lim}_{n\to+\infty} \mathcal{W}_n^* \setminus \mathcal{W}^* = \emptyset$.

Proof. Suppose $\overline{\lim}_{n\to+\infty} \mathcal{W}_n^* \setminus \mathcal{W}^* \neq \emptyset$. So there is at least one $y \in \overline{\lim}_{n\to+\infty} \mathcal{W}_n^* \setminus \mathcal{W}^*$. This leads to the existence of a subsequence $\{n_k\}$ of K such that $y \in \mathcal{W}_{n_k}^* \setminus \mathcal{W}^*$.

- If $y \in \chi_\tau$, then, there is $\overline{y} \in \chi_\tau$, such as $f_i(\overline{y}) < f_i(y), \forall i = 1, \cdots, p$ because $y \notin \mathcal{W}^*$.

  According to Lemma 3.1, $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_n \tau_{n_k}) = 0$ and $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_n \tau_{n_k}) = 0$. Consequently, from a certain rank $k_0 > 0$, as soon as $k \geq k_0$, we have

$$f_i(\overline{y}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k}$$
$$< f_i(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k},$$

$\forall i = 1, \cdots, p$. That is a contradiction about $y \in \mathcal{W}^*_{n_k}$.

- If $y \notin \chi_\tau$ then, according to Lemma 3.1, $\lim\limits_{n\to+\infty}(-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n\tau_n) = +\infty$ therefore $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k}) = +\infty$.

Let $\overline{y} \in \chi_\tau$, we have $\lim\limits_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k}) = 0$.

We deduce that from a certain rank k_0 such as $k > k_0$, we have

$$8f_i(\overline{y}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k}$$
$$< f_i(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k},$$

$\forall i = 1, \cdots, p$.

This contradicts the relation $y \in \mathcal{W}^*_{n_k}$. Hence the Theorem. □

Theorem 3.3. $\lim\limits_{n\to+\infty} \mathcal{W}^*_n \setminus \mathcal{W}^* = \emptyset$.

Proof. According to Theorem 3.1 and Theorem 3.2, we have $\underline{\lim}_{n\to+\infty} \mathcal{W}^*_n \setminus \mathcal{W}^* = \overline{\lim}_{n\to+\infty} \mathcal{W}^*_n \setminus \mathcal{W}^* = \emptyset$. Hence the theorem. □

Theorem 3.4. *Let $\{x^*_n\} \subset \mathcal{W}^*_n, n = 1, 2, \cdots$, a sequence of weakly Pareto optimal solutions of the problem (3.1). If $\{x^*_{n_k}\}$ is a convergence subsequence of $\{x^*_n\}$ and $\lim\limits_{k\to+\infty} x^*_{n_k} \in \chi_\tau$, then $\lim\limits_{k\to+\infty}(-\tau_{n_k} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}) = 0$.*

Proof. suppose that $\lim\limits_{k\to+\infty} \tau_{n_k} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k} \neq 0$, then there exists a subsequence $\{x^*_{n_{k_s}}\}$ of $\{x^*_{n_k}\}$ such as $-\tau_{n_k} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_k}\tau_{n_{k_s}} > \alpha$, with $s = 1, 2, \cdots$, and α is a strict positive number. As $\{x^*_{n_{k_s}}\}$ is a solution of the following multiobjective optimization problem. We have

$$\min\Big\{f_1(x)-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x))+\sigma_{n_{k_s}}\tau_{n_{k_s}},\cdots,f_p(x)-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x))+\sigma_{n_{k_s}}\tau_{n_{k_s}}\Big\} \tag{3.5}$$

there is no $\overline{x}\in\mathbb{R}^n$ such as

$$f_i(\overline{x})-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(\overline{x}))+\sigma_{n_{k_s}}\tau_{n_{k_s}}<f_i(x^*_{n_{k_s}})-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x^*_{n_{k_s}}))+\sigma_{n_{k_s}}\tau_{n_{k_s}},\ i=1,2,\cdots,\ p;s=1,2,\cdots. \tag{3.6}$$

As $\lim\limits_{k\to+\infty}x^*_{n_k}=x^*$, this inequality

$$f_i(x^*)-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x^*))+\sigma_{n_{k_s}}\tau_{n_{k_s}}<f_i(x^*_{n_{k_s}})-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x^*_{n_{k_s}}))+\sigma_{n_{k_s}}\tau_{n_{k_s}},\ i=1,2,\cdots,\ p;s=1,2,\cdots,$$

is not verified. Like $x^*_{n_k}\in\mathcal{W}^*_n$, there is $i_s\in\{1,2,\cdots,p\}$ such as $f_{i_s}(x_{n_{k_s}})\leq f_{i_s}(x^*)$ and

$$\begin{aligned}&f_{i_s}(x_{n_{k_s}})-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x_{n_{k_s}}))+\sigma_{n_{k_s}}\tau_{n_{k_s}}\\ \leq\ &f_{i_s}(x^*)-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x^*))+\sigma_{n_{k_s}}\tau_{n_{k_s}}.\end{aligned}$$

As the set $\{1,2,\cdots,p\}$ is finite, there exists an infinite number of terms of the sequence $\{i_s\}$ such that the terms are all equal in the set $\{1,2,\cdots,p\}$. So, we have for $i_s=1$, we have

$$\begin{aligned}&f_1(x_{n_{k_s}})-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x_{n_{k_s}}))+\sigma_{n_{k_s}}\tau_{n_{k_s}}\\ \leq\ &f_1(x^*)-\tau_{n_{k_s}}\sum_{j=1}^{m}\ln(\tau_{n_{k_s}}-g_j(x^*))+\sigma_{n_{k_s}}\tau_{n_{k_s}},\end{aligned}$$

$s = 1, 2, \cdots$. Like $\tau_{n_k} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}} \tau_{n_{k_s}} > \alpha, s = 1, 2, \cdots$, we have

(3.7) $$f_1(x_{n_{k_s}}) + \alpha \leq f_1(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}} \tau_{n_{k_s}}, s = 1, 2, \cdots .$$

Furthermore, $x^* \in \chi_\tau$, then $\lim_{s\to\infty} \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}} \tau_{n_{k_s}} = 0$, based on the continuity of the function f_1 and $x^*_{n_{k_s}} \to x^*$, when $s \to \infty$, we have $\alpha \leq 0$, that is nonsense because $\alpha > 0$. □

Theorem 3.5. $\underline{\lim}_{n\to+\infty} \mathcal{W}^*_n \subseteq \overline{\lim}_{n\to+\infty} \mathcal{W}^*_n \subseteq \chi_\tau$.

Proof. We have $\underline{\lim}_{n\to+\infty} \mathcal{W}^*_n \subseteq \overline{\lim}_{n\to+\infty} \mathcal{W}^*_n$ or by using Lemma 3.2, we have $\overline{\lim}_{n\to+\infty} \mathcal{W}^*_n \subseteq \chi_\tau$. Hence the result. □

Theorem 3.6. *Let $\{x^*_n\} \subset \mathcal{W}^*_n, n = 1, 2, \cdots$, a sequence of weakly Pareto optimal solutions of the problem (3.1) . If $\{x^*_{n_k}\}$ is a convergence subsequence of $\{x^*_n\}$ and $\lim_{k\to+\infty} x^*_{n_k} \in \chi_\tau$, then $\lim_{k\to+\infty} x^*_{n_k} \in \mathcal{W}^*$.*

Proof. Let $\lim_{k\to+\infty} x^*_{n_k} = x^*$. As $x^* \in \mathcal{W}^*_n$ then, there is no solution $\overline{x} \in \chi_\tau$ such as

(3.8) $$f_i(\overline{x}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} < f_i(x^*_{n_k}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, \forall i = 1, \cdots, p; k = 1, 2, \cdots$$

As $\lim_{k\to+\infty} \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{x})) - \sigma_{n_k}\tau_{n_k} = 0$ and $\lim_{k\to+\infty} \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*_{n_k})) - \sigma_{n_k}\tau_{n_k} = 0$ so for $k \to +\infty$, there is no $f_i(\overline{x}) < f_i(x^*), \forall i = 1, 2, \cdots, p$. Otherwise, if there is a $y \in \chi_\tau$ such as $f_i(y) < f_i(x^*), \forall i = 1, 2, \cdots, p$ then, from a certain rank k_0 such as $k \geq k_0$ we have

$$f_i(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k} < f_i(x^*_{n_k}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, \quad \forall i = 1, \cdots, p; k = 1, 2, \cdots .$$

Which contradicts the inequality (3.8). Hence the theorem. □

3.2.2. *Towards Pareto optimal solutions.*
Note that, any Pareto optimal solution is weakly Pareto optimal but the converse is not true.

Theorem 3.7. $\underline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^* = \emptyset$.

Proof. Suppose that $\underline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^* \neq \emptyset$. So, there is at least one $y \in \underline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^*$. So, from a certain rank n_0 such as $n \geq n_0$, we have $y \in \mathcal{P}_n^* \setminus \mathcal{P}^*$. It follows that $y \in \mathcal{P}_n^*$ and $y \notin \mathcal{P}^*$. Two cases arise:

Case 1: if $y \in \chi_\tau$, then, there exists an $\overline{y} \in \chi_\tau$ such as $f_i(\overline{y}) \leq f_i(y), \forall i = 1, \cdots, p$ and for at least one $k \in \{1, \cdots, p\}$, we have $f_k(\overline{y}) < f_k(y)$. Therefore, from a certain rank n_0, we have the inequality

$$f_i(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$\leq f_i(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n,$$

$\forall i = 1, \cdots, p$, which is true and for at least one $k \in \{1, \cdots, p\}$, we have

$$f_k(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$\leq f_k(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n,$$

which is true. This contradicts $y \in \mathcal{P}_n^*$.

Case 2: if $y \notin \chi_\tau$, then, by Lemma 3.1, $\lim_{n\to+\infty} (-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n) = +\infty$. Let us consider a $\overline{y} \in \chi_\tau$. We have $\lim_{n\to+\infty} (-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n) = 0$. So, from a certain rank $n_0 > 0$ such as $n \geq n_0$, we have:

$$f_i(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$\leq f_i(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n, \forall i = 1, \cdots, p.$$

which is true and for at least one $k \in \{1, \cdots, p\}$ we have

$$f_k(\overline{y}) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(\overline{y})) + \sigma_n \tau_n$$
$$< f_k(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n$$

which is also true. These last two relations are absurd because $y \in \mathcal{P}_n^*$.

Therefore, we can conclude that $\underline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^* \neq \emptyset$. That is absurd hence the theorem. $\square$

Theorem 3.8. $\overline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^* = \emptyset$.

Proof. Suppose that $\overline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^* \neq \emptyset$. So there is at least one $y \in \overline{\lim}_{n\to+\infty} \mathcal{P}_n^* \setminus \mathcal{P}^*$. Thus there exists a subsequence $\{n_k\}$ of K such as $y \in \mathcal{P}_{n_k}^* \setminus \mathcal{P}^*$. So $y \in \mathcal{P}_{n_k}^*$ and $y \notin \mathcal{P}^*$.

Suppose that $y \in \chi_\tau$. Like $y \notin \mathcal{P}^*$ then, it exists $\overline{y} \in \chi_\tau$, such as $f_i(\overline{y}) \leq f_i(y), \forall i = 1, \cdots, p$ and for at least one $k \in \{1, \cdots, p\}$, we have $f_k(\overline{y}) < f_k(y)$.

Also, from Lemma 3.1, $\lim_{k\to+\infty} (-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k} \tau_{n_k}) = 0$ and $\lim_{k\to+\infty} (-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k} \tau_{n_k}) = 0$. So, from a certain rank $k_0 > 0$, as soon as $k \geq k_0$, we have $f_i(\overline{y}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k} \tau_{n_k} < f_i(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k} \tau_{n_k}, \forall i = 1, \cdots, p$, which is true and for at least one $i' \in \{1, \cdots, p\}$, we have $f_{i'}(\overline{y}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k} < f_{i'}(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k}$ which is true. This contradicts the relation $y \in \mathcal{P}^*_{n_k}$.

If $y \notin \chi_\tau$, then, according to Lemma 3.1, $\lim_{n\to+\infty}(-\tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n\tau_n) = +\infty$. Therefore, $\lim_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k}) = +\infty$. let's choose $\overline{y} \in \chi_\tau$, we have $\lim_{k\to+\infty}(-\tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k}) = 0$.

We deduce that from a certain rank k_0 such as $k > k_0$, we have

$$f_i(\overline{y}) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k} \leq f_i(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k},$$

$\forall i = 1, \cdots, p$, which is true and for at least one $i' \in \{1, \cdots, p\}$, $f_{i'}(\overline{y}) - \tau_{n_k} \cdot \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(\overline{y})) + \sigma_{n_k}\tau_{n_k} < f_{i'}(y) - \tau_{n_k} \sum_{j=1}^{m} \ln(\tau_{n_k} - g_j(y)) + \sigma_{n_k}\tau_{n_k}$ is true. This contradicts the relation $y \in \mathcal{P}^*_{n_k}$, hence the theorem. $\square$

Theorem 3.9. *Let $\{x^*_n\} \subset \mathcal{P}^*_n, n = 1, 2, \cdots$, a sequence of Pareto optimal solutions of the problem (3.1). If $\{x^*_{n_k}\}$ is a convergence subsequence of $\{x^*_n\}$ and $\lim_{k\to+\infty} x^*_{n_k} \in \chi_\tau$, then $\lim_{k\to+\infty}(-\tau_{n_k} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}) = 0$.*

Proof. Suppose that $\lim_{k\to+\infty}(-\tau_{n_k} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}) \neq 0$. Then, there exists a subsequence $\{x^*_{k_s}\}$ of $\{x^*_{n_k}\}$ such as $(-\tau_{n_k} \ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}) > \alpha, s = 1, 2, \cdots$, with α a strict positive number.

As $x^*_{n_{k_s}}$ is Pareto optimal of (3.1), then, there is no other solution $\overline{x}$ in χ_τ such as $f_i(\overline{x}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(\overline{x})) + \sigma_{n_{k_s}}\tau_{n_{k_s}} \leq f_i(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}, \forall i = 1, \cdots, p; s = 1, 2, \cdots$ and for at least one $i' \in \{1, \cdots, p\}$, we have $f_{i'}(\overline{x}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(\overline{x})) + \sigma_{n_{k_s}}\tau_{n_{k_s}} < f_{i'}(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}, s = 1, 2, \cdots$.

As $\lim_{k\to+\infty} x^*_{n_k} = x^* \in \chi_\tau$, then the following inequality is not verified:

$$f_i(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}} \le f_i(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}, i = 1, \cdots, p; s = 1, 2, \cdots \tag{3.9}$$

and for at least $i' \in \{1, \cdots, p\}$ we do not have

$$f_{i'}(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}} < f_{i'}(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}, s = 1, 2, \cdots . \tag{3.10}$$

Consider the relation (3.9), for any point $x^*_{n_{k_s}}$, there exists a $i_s \in \{1, 2, \cdots, p\}$ such as

$$f_{i_s}(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}} > f_{i_s}(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}. \tag{3.11}$$

We have an infinity of terms i_s of the sequence $\{i_s\}$ in $\{1, \cdots, p\}$. As $\{1, 2, \cdots, p\}$ is finite, there exists an infinity of terms of $\{i_s\}$ such as every term has the same element in $\{1, 2, \cdots, p\}$.

Suppose there are infinitely many terms i_s, which have the value 1. For convenience, let us define the subsequence $\{x^*_{n_{k_{i_s}}}\}$ such as the relation (3.11) is verified in $\{x^*_{n_{k_s}}\}$ with $i_s = 1$. We have $f_1(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}} > f_{i_s}(x^*_{n_{k_s}}) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}}, s = 1, 2, \cdots$. Since $-\tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*_{n_{k_s}})) + \sigma_{n_{k_s}}\tau_{n_{k_s}} > \alpha, s = 1, 2, \cdots$, we have

$$f_1(x^*) - \tau_{n_{k_s}} \sum_{j=1}^{m} \ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}} > f_{i_s}(x^*_{n_{k_s}}) + \alpha, s = 1, 2, \cdots .$$

As $x^* \in \chi_\tau$, $\lim\limits_{s\to+\infty}\Big(-\tau_{n_{k_s}}\sum\limits_{j=1}^{m}\ln(\tau_{n_{k_s}} - g_j(x^*)) + \sigma_{n_{k_s}}\tau_{n_{k_s}}\Big) = 0$. Moreover, as the f_i are continuous on χ_τ and $\lim\limits_{s\to+\infty} x^*_{n_{k_s}} = x^*$ then, $\lim\limits_{s\to+\infty} f_1(x^*_{n_{k_s}}) = f_1(x^*)$.

So, $f_1(x^*) > f_1(x^*) + \alpha$ hence $0 > \alpha$. Which is absurd because $\alpha > 0$.

By doing a similar reasoning with the relation (3.10), we show that $0 \geq \alpha$. Which is also absurd, hence the Theorem. □

Theorem 3.10. *Let $\{x^*_n\} \subset \mathcal{P}^*_n, n = 1, 2, \cdots$, a sequence of Pareto optimal solutions of the problem (3.1). If $\{x^*_{n_k}\}$ is a convergence subsequence of $\{x^*_n\}$ and $\lim\limits_{k\to+\infty} x^*_{n_k} \in \chi_\tau$, then $\lim\limits_{k\to+\infty} x^*_{n_k} \in \mathcal{P}^*$.*

Proof. Suppose $x^* = \lim\limits_{k\to+\infty} x^*_{n_k} \in \chi_\tau$. As $x^*_{n_k} \in \mathcal{P}^*_{n_k}$, then there is no $\overline{x}$ in χ_τ such that

$$f_i(\overline{x}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} \leq f_i(x^*_{n_k}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, i = 1, 2, \cdots, p; k = 1, 2, \cdots \tag{3.12}$$

and for at least one $i' \in \{1, \cdots, p\}$, we have

$$f_{i'}(\overline{x}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} < f_{i'}(x^*_{n_k}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, i = 1, 2, \cdots, p; k = 1, 2, \cdots \tag{3.13}$$

As $\overline{x} \in \chi_\tau$ and $x^*_{n_k} \in \chi_\tau$, then, according to Lemma 3.1, $\lim\limits_{k\to+\infty}\Big(-\tau_{n_k}\sum\limits_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k}\Big) = 0$, and $\lim\limits_{k\to+\infty}\Big(-\tau_{n_k}\sum\limits_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}\Big) = 0$.

As the f_i are continuous and $x^* = \lim\limits_{k\to+\infty} x^*_{n_k}$, then, there is no $\overline{x} \in \chi_\tau$ such as for k tending to infinity, $f_i(\overline{x}) \leq f_i(x^*), \forall i = 1, \cdots, p$ and for at least one $i' \in \{1, \cdots, p\}$; $f_{i'}(\overline{x}) < f_{i'}(x^*)$.

If we assume that, there is an $\overline{x} \in \chi_\tau$ such as $f_i(\overline{x}) \leq f_i(x^*), \forall i = 1, \cdots, p$ and for at least one $i' \in \{1, \cdots, p\}$; $f_{i'}(\overline{x}) < f_{i'}(x^*)$ then, from a certain rank k_0 such as $k \geq k_0$, we will have:

$$f_i(\overline{x}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} \leq f_i(x^*_{n_k}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, i = 1, 2, \cdots, p; k = 1, 2, \cdots \tag{3.14}$$

and for at least one $i' \in \{1, \cdots, p\}$,

$$f_{i'}(\overline{x}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(\overline{x})) + \sigma_{n_k}\tau_{n_k} < f_{i'}(x^*_{n_k}) - \tau_{n_k}\sum_{j=1}^{m}\ln(\tau_{n_k} - g_j(x^*_{n_k})) + \sigma_{n_k}\tau_{n_k}, i = 1, 2, \cdots, p; k = 1, 2, \cdots . \tag{3.15}$$

The relations (3.14) and (3.15) contradict (3.12) and (3.13), which show that $x^*_{n_k} \in \mathcal{P}^*_n$, hence the Theorem. □

Now, we will prove that any Pareto optimal solution of problem (2.1) is Pareto optimal of problem (3.1).

Theorem 3.11. $\mathcal{P}^*_n = \mathcal{P}^*$.

Proof. Let us show that $\mathcal{P}^*_n \subseteq \mathcal{P}^*$ and let $x^* \in \mathcal{P}^*_n$. So, there is no $y \in \chi_\tau$ such as $f_i(y) - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(y)) + \sigma\tau_n \leq f_i(x^* - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(x^*)) + \sigma_n\tau_n, \forall i = 1, \cdots, p$ and for at least one $q \in \{1, \cdots, p\}$, we have $f_q(y) - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(y)) + \sigma_n\tau_n <$ $f_q(x^*) - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(x^*)) + \sigma_n\tau_n$.

Suppose that $x^* \notin \mathcal{P}^*$, then, there exists $y \in \chi_\tau$ such as the inequalities $f_i(y) \leq f_i(x^*), \forall i = 1, \cdots, p$ and for at least one $q \in \{1, 2, \cdots, p\}$, $f_q(y) < f_q(x^*)$. According to Lemma 3.2, as $y \in \chi_\tau$, then $\lim_{n\to+\infty}\tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(y)) - \sigma_n\tau_n = 0$. Likewise, $\lim_{n\to+\infty}\tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(x^*)) - \sigma_n\tau_n = 0$. So, the inequality $f_i(y) - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(y)) + \sigma_n\tau_n \leq f_i(x^* - \tau_n\sum_{j=1}^{m}\ln(\tau_n - g_j(x^*)) + \sigma_n\tau_n, \forall i = 1, \cdots, p$

and for at least one $q \in \{1, \cdots, p\}$, we have $f_q(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n < f_q(x^*) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x^*)) + \sigma_n \tau_n$ is true from a certain rank. Which contradicts the fact that $x^* \in \mathcal{P}_n^*$. Therefore, $x^* \in \mathcal{P}^*$.

Now let us show that $\mathcal{P}^* \subseteq \mathcal{P}_n^*$. Let $x^* \in \mathcal{P}^*$, then, there is no $y \in \chi_\tau$ such as $f_i(y) \leq f_i(x^*), \forall i = 1, \cdots, p$ and for at least one $q \in \{1, 2, \cdots, p\}$,$f_q(y) < f_q(x^*)$.

Suppose there is a $y \in \mathcal{P}_n^*$ such as $f_i(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n \leq f_i(x^* - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x^*)) + \sigma \tau_n, \forall i = 1, \cdots, p$ and for at least one $q \in \{1, \cdots, p\}$, we have $f_q(y) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) + \sigma_n \tau_n < f_q(x^*) - \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x^*)) + \sigma \tau_n$. According to Lemma 3.2, we have $\lim_{n \to +\infty} \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) - \sigma_n \tau_n = 0$ and $\lim_{n \to +\infty} \tau_n \sum_{j=1}^{m} ln(\tau_n - g_j(x^*)) - \sigma_n \tau_n = 0$ because y and $x^* \in \chi_\tau$. According to Lemma 3.2, we have $\lim_{n \to +\infty} \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(y)) - \sigma_n \tau_n = 0$ and $\lim_{n \to +\infty} \tau_n \sum_{j=1}^{m} \ln(\tau_n - g_j(x^*)) - \sigma_n \tau_n = 0$ because y and $x^* \in \chi_\tau$. Therefore, $f_i(y) \leq f_i(x^*), \forall i = 1, \cdots, p$ and for at least one $q \in \{1, 2, \cdots, p\}$,$f_q(y) < f_q(x^*)$. Which is absurd because $x^* \in \mathcal{P}^*$, hence the theorem. □

4. Conclusion

In this work, we have proposed an extension of the logarithm barrier penalty function for solving nonlinear multiobjective optimization problems with inequality constraints. Through some theorems and proofs, we have established the theoretical foundations of convergence to Pareto optimal solutions for our new approach. It was shown in our work that the logarithm barrier penalty function can be used to get Pareto optimal solutions in cases of multiobjective optimization with inequality constraints. It is possible to highlight the effectiveness of this new technique when applied to practical cases or to test problems for the determination of optimal Pareto solutions.

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