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ALIENOR METHOD APPLIED IN STATISTICS: OPTIMIZATION OF KATZ'S AND COM-POISSON'S LIKELIHOOD FUNCTIONS

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ABSTRACT. In this paper, we are interested in a new approach to optimizing the likelihood function of a distribution using the Alienor method. Indeed, the Alienor method is an optimization technique that allows to reduce a multivariate optimization problem to a single variable optimization problem that is easy to solve numerically. For this purpose, we consider the Katz and COM-Poisson distributions, whose maximization of their likelihood functions requires numerical methods. In a first step, we used real data, and, in a second step, we performed a simulation study. The results obtained are satisfactory, and the Alienor method proves to be very interesting in statistics, both for its simplicity and for its performance in converging to the absolute optimum.

1. INTRODUCTION

One of the key steps in statistical inference is the estimation of the model parameters, and one of the most popular parametric estimation methods is the maximum likelihood method. Given an $n-$ sample $(y_1, \dots, y_n)$ independently and identically distributed, the maximum likelihood method consists in maximizing the likelihood function of a parametric model given by:

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$$L(\theta|y_1,\ldots,y_n) = \prod_{i=1}^{n} p(y_i|\theta),$$

where θ is the vector of parameters belonging to the parameter space Θ and $p(y|\theta)$ the parameterized probability distribution by θ. The maximum sought is reached for $\widehat{\theta}$ such that:

$$\widehat{\theta} = \arg\max_{\theta\in\Theta} L(\theta|y_1,\ldots,y_n). \tag{1.1}$$

Most often, the logarithm of the likelihood function is maximized, $l(\theta) = \log L(\theta|y_1,\ldots,y_n)$, because it is easier to use analytically. The problems of existence and uniqueness as well as the asymptotic behavior of estimators of the maximum likelihood given by (1.1) have been widely and variously studied in the literature [18]. For many models, the problem (1.1) cannot be solved analytically, and therefore, numerical methods are used. Several numerical algorithms have been proposed in the literature, in particular, the Newton-Raphson algorithm and the EM algorithm, as well as their extensions [17, 18].

The problem (1.1) is one of the global optimization problems in operations research. As for (1.1), maximizing or minimizing an objective function leads, in general, to a numerical solution. And generally, global optimization problems cannot be solved efficiently by classical optimization techniques: this is due to the fact that most classical methods only converge towards local minima [11]. Among the numerical methods of global optimization proposed in the literature is the Alienor method. Developed in the 1980s by Cherruault and Guillez [3,11], the Alienor method consists of reducing a multivariate objective function to a univariate objective function using a reductive transformation with an optimization preserving operator. In the literature, statistical work using this method remains almost nonexistent. In this paper, we are interested in a new approach: maximizing the likelihoods of the Katz [10] and Conway-Maxwell-Poisson [5] distributions (in short, COM-Poisson or CMP) using the Alienor method. Indeed, the Katz and COM-Poisson distributions are two-parameter distributions that include, in particular, the Poisson distribution by taking into account the situations of overdispersion and underdispersion. These two distributions have been widely and variously studied in the literature and have been used as a basis to develop several families of distributions [1, 7, 9, 14, 19]. The problem (1.1) corresponding to these distributions cannot be solved analytically and therefore requires numerical methods.

The remainder of this paper is presented as follows. In Section 2, the Katz and COM-Poisson distributions are presented. In Section 3, the Alienor method is described in a concise way. In Section 4, we present the results obtained. First, we make use of the real data, and second, we perform a simulation study considering the situations of overdispersion and underdispersion and simulating samples of different sizes for each situation and each distribution. The conclusion and perspectives are presented in Section 5.

2. Katz's and COM-Poisson's distributions

In this section, we focus on the Katz and COM-Poisson distributions by presenting their probability mass functions and the corresponding likelihood functions.

2.1. **Katz's distribution.**

The Katz distribution [10] is defined from to the successive probability ratios:

$$\frac{p(y+1)}{p(y)} = \frac{\lambda+\beta y}{y+1},\ y = 0, 1, \dots, \tag{2.1}$$

with $p(0) \neq 0$ and $p(y) = P(Y = y)$, where $\lambda > 0$ and $\beta < 1$, it is understood that if $\lambda + \beta y < 0$ then $p(y) = 0$ for $y = 1, 2, \dots$ [1]. The probability mass function pmf corresponding to (2.1) is given by [1]:

$$p(y) = \begin{cases} \dfrac{\lambda^y}{y!}e^{-\lambda} & \text{if } \beta = 0, \\ \dfrac{(\lambda/\beta)_y \beta^y}{y!}(1-\beta)^{\lambda/\beta} & \text{otherwise}, \end{cases} \tag{2.2}$$

$y = 0, 1, \dots,$ where $(\alpha)_y$ is the Pochhammer symbol and defined to be $(\alpha)_y = \alpha(\alpha+1)\dots(\alpha+y-1)$ for $y = 0, 1, \dots,$ and α any real number with $(\alpha)_0 = 1$. This distribution is a good way to unif Poisson, binomial, and negative binomial distributions when $\beta = 0,\ \beta < 0$ and $\beta > 0$, respectively [1].

Let put $\theta = (\lambda, \beta)$ and given an $n-$sample $y = (y_1, \dots, y_n)$ and for $\beta \neq 0$, the log-likelihood function $l(\theta)$ of the variable Y Katz distributed is given by [14]:

$$l(\theta) = \frac{n\lambda}{\beta}\log(1-\beta) + \sum_{i=1}^{n}\sum_{k=1}^{y_i}\log\left[\lambda + \beta(k-1)\right] - n\overline{\log(y!)}, \tag{2.3}$$

with the convention $\sum_{k=1}^{0} = 0$ and where $\overline{\log(y!)} = \frac{1}{n}\sum_{i=1}^{n}\log(y_i!)$.

2.2. **COM-Poisson's distribution.** The pmf of the CMP distribution is given by [5]:

$$p(y) = \frac{\lambda^y}{y!^\nu}\frac{1}{Z(\lambda,\nu)}, \quad y = 0, 1, \ldots, \tag{2.4}$$

where,

$$Z(\lambda,\nu) = \sum_{j\geq 0}\frac{\lambda^j}{j!^\nu}, \quad \lambda > 0,\ \nu \geq 0,$$

is the normalizing constant. This distribution is a good way to unif Poisson, Bernoulli and geometric distributions when $\nu = 1$, $\nu \to \infty$, and $\nu = 0$ and $0 < \lambda < 1$, respectively [7, 19].

Let put $\theta = (\lambda, \nu)$ and given an $n-$sample $y = (y_1, \ldots, y_n)$, the log-likelihood function $l(\theta)$ of the variable Y CMP distributed is given by [19]:

$$l(\theta) = n\overline{y}\log\lambda - n\nu\overline{\log y!} - n\log Z(\lambda,\nu), \tag{2.5}$$

where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

3. Alienor method

In this section, we present the Alienior method by describing the principle of the method and giving the algorithm for using said method.

3.1. **Principe.** Consider the following optimization problem [12]:

$$Glob \min_{(x_1,\ldots,x_p)\in\prod_{k=1}^{p}[a_k, b_k]} f(x_1, \ldots, x_p), \tag{3.1}$$

where f is a continuous function. The Alienor methode consist to substitute the variables $x_k,\ k=\overline{1,p}$ by the reducing transformation [11, 12]:

$$x_k = \frac{1}{2}\left[(b_k - a_k)h_k(x) + (b_k + a_k)\right], \quad x \in [-1,1]. \tag{3.2}$$

Thus, the multidimensional optimization problem (3.1) is reduced to the unidimensional optimization problem (3.3):

$$Glob \min_{x\in\ [-1,1]} f^*(x), \tag{3.3}$$

where $f^*(x) = f(x_1,\ldots,x_p)$ using (3.2).

Several reductive transformations have been proposed; we will be interested in the following reductive transformations:

(i) Mora transformation [4]:

$$\begin{cases} h_1(x) = x, \\ h_k(x) = 1 - \cos\left(m^{k-1}\pi x\right),\ k=\overline{2,p},\ m=2 \text{ or } 3; \end{cases}$$

(ii) Konfé-Cherruault transformation [11, 12]:

$$h_k(x) = \cos(\omega_k x + \varphi_k),\ k=\overline{1,p},$$

with $\omega_k = 100 + 0.0005 \times k$ and $\varphi_k = 1 + 0.0000005(k-1)$.

3.2. **Algorithm.** The problem given by (3.3) is easy to solve numerically, because it is an optimization problem of one dimension [2].

To improve the quality of the estimator, we combine the following operator, called the new optimization-preserving operator, with the Alienor method. (in short, OPO*) [11–13, 16]:

$$T^{\varepsilon}_{f^*}(x) = \frac{f^*(x) - f^*(x_0) + |f^*(x) - f^*(x_0)|}{2} + \frac{1}{\varepsilon} \times H(f^*(x) - f^*(x_0)),$$

with x_0 an arbitrary point of $[-1,\ 1]$, ε a small positive real and H is the Heaviside function. Solving the optimization problem of (3.3) amounts to solving the equation $T^{\varepsilon}_{f^*}(x) = 0$.

The steps of the algorithm are as follows [13]:

(i) Step 0: Initialization
An initial point x_0, and a constant ε are given. Then we have:

$$T^{\varepsilon}_{f^*(x_0)}(x) = \frac{f^*(x) - f^*(x_0) + |f^*(x) - f^*(x_0)|}{2} + \frac{1}{\varepsilon} \times H(f^*(x) - f^*(x_0)).$$

(ii) Step 1: Step calculation
Solve $T^{\varepsilon}_{f^*(x_i)}(x) = 0$.
Let S_{x_i} be a set defined by:

$$S_{x_i} = \left\{x \in [-1,1] : T^{\varepsilon}_{f^*(x_i)}(x) = 0\right\}.$$

If $S_{x_i} = \{x^*\}$,
then x^* is a global minimizer of f^*,
end;
otherwise go to Step 2.

(iii) Step 2: Update $T^{\varepsilon}_{f^*}$
For $i = 1, 2, \ldots, p, \ldots$, consider $x_i \in S_{x_{i-1}} = \left\{x \in [-1,1] : T^{\varepsilon}_{f^*(x_{i-1})}(x) = 0\right\}$ and build:

$$T^{\varepsilon}_{f^*(x_i)}(x) = \frac{f^*(x) - f^*(x_i) + |f^*(x) - f^*(x_i)|}{2} + \frac{1}{\varepsilon} \times H(f^*(x) - f^*(x_i)),$$

then go to Step 1.

Remark 3.1. *Theorem 8 in [13] ensures that if S_{x_i} contains a unique element, it is the solution of the global minimization problem* (3.3).

4. Results

We use the Alienor method in this section to maximize the likelihood functions of the Katz' and COM-Poisson' distributions. For Katz' distribution, we use the two reduction transformations, while for the COM-Poisson distribution, we use only the Mora transformation. For Katz' distribution, we use the two reduction transformations while for the COM-Poisson distribution, we use only the Mora transformation. The Alienor method algorithm was programmed under the Maplesoft language [12] and the results are compared with those of the extensions of the Newton-Raphson algorithm using the maxLik package for the r statistical environment [8].

Moreover, the Alienor method as presented deals with minimization problems, and a maximization problem will be reduced to a minimization problem by the relation:

$$\max(f) = -\min(-f).$$

4.1. **Maximization of the likelihood of the Katz's distribution.**

4.1.1. *Objetive function.* For Mora transformation, we have:

$$\begin{cases} x_1 = \lambda = \frac{1}{2}[(b_1 - a_1)x + (b_1 + a_1)], \\ x_2 = \beta = \frac{1}{2}[(b_2 - a_2)(1 - \cos(m\pi x)) + (b_2 + a_2)], \quad m = 2 \text{ or } 3, \end{cases}$$

and applying this transformation in (2.3), we obtain the following objective function:

$$\begin{aligned} f^*(x) = & \frac{n[(b_1 - a_1)x + b_1 + a_1]}{(b_2 - a_2)(1 - \cos(m\pi x)) + b_2 + a_2} \\ & \cdot \log\left(1 - \frac{1}{2}[(b_2 - a_2)(1 - \cos(m\pi x)) + (b_2 + a_2)]\right) \\ & + \sum_{i=1}^{n} \sum_{k=1}^{y_i} \log\left[(b_1 - a_1)x + b_1 + a_1 + (k-1)[(b_2 - a_2)(1 - \cos(m\pi x))\right. \\ & \left. + b_2 + a_2]\right] - n\left[1 - \overline{\delta_0(y)}\right] \log 2 - n\overline{\log(y!)}, \end{aligned}$$

where $\overline{\delta_0(y)} = \frac{1}{n}\sum_{i=1}^{n} \delta_0(y_i)$.

For Konfé-Cherruault transformation, we have:

$$\begin{cases} x_1 = \lambda = \frac{1}{2}[(b_1 - a_1)\cos(\omega_1 x + \varphi_1) + (b_1 + a_1)], \\ x_2 = \beta = \frac{1}{2}[(b_2 - a_2)\cos(\omega_2 x + \varphi_2) + (b_2 + a_2)], \end{cases}$$

and

$$f^*(x) = \frac{n[(b_1 - a_1)\cos(\omega_1 x + \varphi_1) + b_1 + a_1]}{(b_2 - a_2)\cos(\omega_2 x + \varphi_2) + b_2 + a_2}$$

$$\times \log\left(1 - \frac{1}{2}[(b_2 - a_2)\cos(\omega_2 x + \varphi_2) + (b_2 + a_2)]\right)$$
$$+ \sum_{i=1}^{n}\sum_{k=1}^{y_i} \log\left[(b_1 - a_1)\cos(\omega_1 x + \varphi_1) + b_1 + a_1\right.$$
$$\left. +(k-1)[(b_2 - a_2)\cos(\omega_2 x + \varphi_2) + b_2 + a_2]\right]$$
$$- n\left[1 - \overline{\delta_0(y)}\right]\log 2 - n\overline{\log(y!)}.$$

4.1.2. *Empirical results.*

Empirical data set 1. Table 1 contains the data set and these data show the distribution of the number of accidents among machine operators in a fixed time period [6]. Table 2 presents the obtain results. We observe that the likelihood is better optimized by the Alienor method with the Konfé-Cherruault transformation.

TABLE 1. Number accidents for machine operators [6]

Observation	0	1	2	3	4	5	Total
No. of accidents	447	132	42	21	3	2	647

TABLE 2. Results of the number accidents for machine operators

	MLE	AM	
		MT	KCT
λ	0.4652241	0.4657328	0.3396944
β	0.5377934	0.5377705	1.4×10^{-9}
$l(\theta)$	-592.2670976	-592.2672266	-453.8482754

MLE: maximum likelihood estimation
AM: Alienor method
MT: Mora transformation
KCT: Konfé-Cherruault transformation

Empirical data set 2. Table 3 contains the data set and these data show the distribution of the Number of weevil eggs laid per bean [15]. Table 4 presents the obtain results. As for the first data set, we observe that the likelihood is better optimized by the Alienor method with the Konfé-Cherruault transformation.

TABLE 3. Number of weevil eggs laid per bean [15]

Observation	0	1	2	3	4+	Total
No. of weevil eggs	159	64	13	4	0	240

TABLE 4. Results of the number of weevil eggs laid per bean

	MLE	AM	
		MT	KCT
λ	0.42500	0.49469	0.33969
β	0.06737	0.25882	1.4×10^{-9}
$l(\theta)$	205.2142	-207.0134453	-176.1779512

4.1.3. *Simulation results.* We simulated samples of size $n = 50,\ 150,\ 500,\ 1000,$ 10000, following the Katz distribution. Two scenarios were investigated based on the values of the dispersion parameter $beta$, -0.05, and 0.5, which correspond to the underdispersion and overdispersion for $lambda = 1.5$, respectively. The results are presented in Tables 5 and 6, respectively for $\beta = -0.05$ and $\beta = 0.5$. We observe that the likelihood is better optimized by the Alienor method with the Konfé-Cherruault reductive transformation in the case of underdispersion and slightly better optimized by the habituated approach over the Mora transformation in the case of overdispersion.

Table 5: Simulation for $\lambda = 1.5$ and $\beta = -0.05$

Run	n	Optimum	MLE	AM	
				MT	KCT
Run 1	50	λ	1.6234	1.62517	1.6186
		β	-0.2684	-0.26887	-0.2679
		$l(\theta)$	-69.44254	-69.44255	-69.44281
Run 2	150	λ	1.61861	1.61353	1.3731
		β	-0.06484	-0.05813	-4.43×10^{-8}
		$l(\theta)$	-228.3376	-228.3405	-228.2348
Run 3	500	λ	1.42144	1.43612	1.3731
		β	-0.02850	-0.03800	-4.43×10^{-8}
		$l(\theta)$	-742.3415	-742.3527	-737.8035

Run 4	1000	λ	1.63023	1.46599	1.3731
		β	-0.13369	-0.02895	-4.43$\times 10^{-8}$
		$l(\theta)$	-1477.304	-1479.6198	-1473.2371
Run 5	10000	λ	1.52042	1.47323	1.3731
		β	-0.05817	-0.02747	-4.43$\times 10^{-8}$
		$l(\theta)$	-14973.57	-14975.7983	-14903.0797

TABLE 6. Simulation for $\lambda = 1.5$ and $\beta = 0.5$

Run	n	Optimum	MLE	AM	
				MT	KCT
Run 1	50	λ	1.4344	1.438836	1.4362
		β	0.3709	0.37018	0.3707
		$l(\theta)$	-96.18827	-96.18845	-96.18831
Run 2	150	λ	1.56678	1.57642	1.5176
		β	0.51631	0.51464	0.5238
		$l(\theta)$	-333.667	-333.668	-333.715
Run 3	500	λ	1.32801	1.42616	1.3148
		β	0.54894	0.51991	0.5460
		$l(\theta)$	-1086.039	-1086.536	-1086.128
Run 4	1000	λ	1.51956	1.59452	1.5318
		β	0.49542	0.47473	0.4889
		$l(\theta)$	-2160.233	-2160.702	-2160.272
Run 5	10000	λ	1.479603	1.47639	1.5178
		β	0.501480	0.50320	0.4870
		$l(\theta)$	-21530.34	-21530.367	-21532.002

4.2. **Maximization of the likelihood of the COM-Poisson's distribution.**

4.2.1. *Objective function.* Consider the Mora transformation:

$$\begin{cases} x_1 = \lambda = \dfrac{1}{2}[(b_1 - a_1)x + (b_1 + a_1)], \\ x_2 = \nu = \dfrac{1}{2}[(b_2 - a_2)(1 - \cos(m\pi x)) + (b_2 + a_2)], \quad m = 2 \text{ or } 3, \end{cases}$$

and applying this transformation in (2.5), we obtain the following objective function:

$$f^*(x) = n\overline{y}\log[(b_1 - a_1)x + b_1 + a_1] - \frac{n}{2}[(b_2 - a_2)(1 - \cos(m\pi x)) + b_2 + a_2]\overline{\log y!} - n\overline{y}\log 2 - n\log Z(x),$$

where

$$Z(x) = \sum_{j\geq 0} \frac{2^{-j}[(b_1 - a_1)x + (b_1 + a_1)]^j}{j!^{2^{-1}[(b_2 - a_2)(1 - \cos(m\pi x)) + (b_2 + a_2)]}}.$$

4.2.2. *Empirical results.*

Empirical data set 1. Table 7 contains the data set and the data consist of the number of death notices of women 80 years of age and older, appearing in the London Times on each day for three consecutive year [7]. Table 8 presents the obtain results. For this data set, we observe that the likelihood is slightly better optimized by the maximum likelihood method.

TABLE 7. Death notice data of London times [7]

Observation	0	1	2	3	4	5	6	7	8	9	Total
No. of death notices	162	267	271	185	111	61	27	8	3	1	1096

TABLE 8. Results of the death notice data of London times

	MLE	MT
λ	1.66024	1.66506
ν	0.74983	0.74956
$l(\theta)$	-1990.143	-1990.15809

Empirical data set 2. Table 9 contains the data set on chromosome interchanges induced by x-ray irradiation [9]. Table 10 presents the obtain results. We observe that the Alienor method slightly optimizes the likelihood for these data.

TABLE 9. Numbers of cells with k interchanges [9]

Observation	0	1	2	3+	Total
No. of cells	2278	273	15	0	2566

TABLE 10. Results of the numbers of cells with k interchanges

	MLE	MT
λ	0.120279	0.120281
ν	1.248943	1.249104
$l(\theta)$	-960.49907	-960.4906

4.2.3. *Simulation results.* As for the Katz distribution, we simulated samples of size $n = 50,\ 150,\ 500,\ 1000,\ 10000$, following the COM-Poisson distribution. Two scenarios were also investigated based on the values of the dispersion parameter ν, 0.4 and 2, which correspond to the overdispersion and underdispersion, respectively, for the same value of $\lambda = 1.5$. Tables 11 and 12 show the results for $\nu = 0.4$ and $\nu = 2$, respectively. We see that the likelihood is better optimized with the Alienor method in the case of overdispersion and better optimized by the habituated approach in the case of underdispersion.

TABLE 11. Simulation for $\lambda = 1.5$ and $\nu = 0.4$

Run	n	Optimum	MLE	MT
Run 1	50	λ	1.4503	1.2438
		ν	0.3525	0.76480
		$l(\theta)$	-117.9516	-109.7439
Run 2	150	λ	1.40633	1.3129
		ν	0.33412	0.86040
		$l(\theta)$	-354.7301	-349.9674
Run 3	500	λ	1.62349	0.92710
		ν	0.45479	0.59697
		$l(\theta)$	-1125.764	-1132.295
Run 4	1000	λ	1.73126	1.63091
		ν	0.48590	0.44772
		$l(\theta)$	-2258.869	-2260.242
Run 5	10000	λ	1.529907	1.513856
		ν	0.413752	0.4072065
		$l(\theta)$	-22721.08	-22721.83

TABLE 12. Simulation for $\lambda = 1.5$ and $\nu = 2$

Run	n	Optimum	MLE	MT
Run 1	50	λ	1.4284	1.50011
		ν	2.0793	2.00000
		$l(\theta)$	-55.72776	-56.24239
Run 2	150	λ	1.5704	1.50017
		ν	2.2617	2.00000
		$l(\theta)$	-164.8709	-164.7383
Run 3	500	λ	1.6381	1.50026
		ν	2.1231	2.00006
		$l(\theta)$	-570.6072	-571.4307
Run 4	1000	λ	1.5391	1.51342
		ν	2.1690	2.02839
		$l(\theta)$	-1114.468	-1115.942
Run 5	10000	λ	1.49753	1.50578
		ν	2.00035	2.00538
		$l(\theta)$	-11465.23	-11465.68

Remark 4.1. *It is assigned that we did not obtain satisfactory results with the Konfé-Cherruault transformation for the COM-Poisson distribution. This can be explained by the fact that the COM-Poisson distribution admits a non-explicit normalization constant, unlike the Katz distribution. Hence the reason to present only the results of the Mora transformation for the COM-Poisson distribution.*

5. CONCLUSION AND PERSPECTIVES

The Alienor method is an interesting optimization technique that allows for reaching the global optimum even for the most complex systems with a high number of variables. Due to its simplicity, its application in statistics also makes it competitive in comparison with the complex algorithms existing in the field. The obtained results let us predict, in a more or less optimistic way, that the use of the Alienor method in statistics will be met with great interest and success.

Like any method, the Alienor method has some drawbacks. The inadequacy of the choice of the bounds that delimit the domain of the parameters makes the value of the likelihood function explode at the optimum. This is not negligible in statistics. This opens perspectives for improving the said method to avoid the explosion of this function. The coupling with another estimation method, such as the method of moments, could better guide the choice of the bounds, especially since the estimators of moments are sometimes used as initial values for the maximum likelihood estimators. Also, we plan to program the Alienor method under the statistical environment r in order to make it more accessible both to those interested in statistics as a decision support tool and to researchers.

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