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SOLVING PARTIAL DIFFERENTIAL EQUATIONS MODELLING SURFACE FLOWS BY THE REDUCED DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT. The aim of this work is to find exact solutions of the non-linear partial differential equations describing the motion of Newtonian fluids at the surface. The reduced differential transform method is used to find the exact solutions of these equations. This method produces an algorithm that favours rapid convergence of the problem towards the exact solution sought.

1. INTRODUCTION

Several equations or systems of partial differential equations model these environmental phenomena. In fluid mechanics, the Navier-Stokes equations are non-linear partial differential equations that describe the motion of Newtonian fluids. A fluid can be a liquid or a gas. These equations are generally non-linear. It should be stressed that it is difficult to solve or find the exact solutions of these equations. However, as approximate solutions, they can often be used to help understand natural phenomena such as ocean currents, the movement of air masses in the atmosphere, the behaviour of atmospheric pressure in built-up areas, the physical situation of bridges under the action of the wind for architects and engineers, or that of planes, trains or high-speed cars for their design offices,

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as well as the flow of water in a pipe and many other phenomena involving the flow of various fluids... [4, 13, 17, 20]. In mathematics, non-linearity complicates matters. In physics, too, the difficulty arises. For the term non-linearity translates into the complexity of the physical phenomena being described. This difficulty of resolution partly affects the analyses or descriptions of the phenomena modeled. In the current context of climate change, a number of serious problems have arisen. These situations have become recurrent, particularly in countries where population growth is considerable. These include deforestation, the presence of many untreated landfill sites, and uncontrolled or inappropriate house-building and urbanization not adapted to current requirements. These waste dumps generate gases that pollute the atmosphere because they are not properly treated. These gas mixtures rise into the atmosphere, causing turbulence and many other situations. The silting-up of rivers, roads and plantations.

Here, we're concerned with flows and transport. We're interested in Reduced Differential Transformation method in the Navier-Stokes equations [15, 19]. In fluid mechanics, the Navier-Stokes equations are nonlinear partial differential equations that describe the motion of Newtonian fluids (i.e. gases and most liquids) [3]. As stated above, the search for exact solutions to these equations modeling a fluid as a continuous fluid as a continuous medium with a single phase is difficult. Our work is motivated by the search for the exact solution of these nonlinear partial differential equations and the problems problems of handling the nonlinear terms. terms. The general objective is to determine the exact solutions of partial differential equations when they exist. The specific The specific objective is to determine the exact solutions of the Navier-Stokes equations in 2D and 3D using the Reduced Differential Transformation transform (RDTM) method.

In the following, the method will be presented, followed by a search for the exact solutions to the various problems selected, and then a conclusion.

2. Description of the Reduced Differential Transformation Method

The Reduced Differential Transformation Method (RDTM) was first proposed by the Turkish mathematician Yildiray Keskin [10, 11]. This method is applicable to a large class if it exists. After Yildiray Keskin and Oturanc [11], The

RDTM has also been used by many authors to obtain analytical approximate and in some cases exact solutions to nonlinear equations. Several types of nonlinear equations have had their different exact solutions easily obtained. We can quote the nonlinear Voltera partial integro-differential equation, the Telegraph equation The inhomogeneous nonlinear wave equation. For more details, we can refer [1, 3, 5, 7–12]. Nevertheless, now suppose that function of two variables $u(x,t)$ which is analytic and k-times continuously differentiable with respect to space x in the domain of our interest [3, 9, 16]. Suppose that we can consider this function in this form: $u(x,t) = f(x)g(t)$. Based on the properties of differential transform, function can be represented as:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(i)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{2.1}$$

where the function $U_k(x)$ is called t-dimensional spectrum function the of $u(x,t)$.

If the function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0}, \qquad k \in \mathbb{N}, \tag{2.2}$$

where the t–dimensional spectrum function $U_k(x)$ is the transformed function. The differential inverse transform of $U_k(x)$ is determined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\left(t-t_0\right)^k = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0}\left(t-t_0\right)^k. \tag{2.3}$$

In fact, the function $u(x,t)$ can written in a finite series as follows,

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)\left(t-t_0\right)^k. \tag{2.4}$$

n is order of approximate where solution.

Therefore, the exact solution of the problem is given by

$$u(x,t) = \lim_{n\to\infty}\tilde{u}_n(x,t). \tag{2.5}$$

The details for the proper understanding of the reduced differential transformation method are well explained by Keskin who is the author [10, 11]. Many researchers have also contributed to facilitate the understanding and use of this rich method [1,3,8,9,12].

To illustrate the basic concepts of the RDM, consider the following nonlinear partial differential equation written in an operator form:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t); \tag{2.6}$$

with initial condition:

$$u(x,0) = f(x). \tag{2.7}$$

According to the RDTM, the iteration formula can be constructed as follows [1,3,8,10–12]

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x). \tag{2.8}$$

Some basic essential properties of the two-dimensional reduced differential transform are presented in Table below [1,3,8,10–12].

TABLE 1. Error table for the "Cell-Centered" method for volumes $N = 4, 8, 12, 16$.

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r u(x,t)}{\partial t^r}$	$W_k(x) = (k+1)\cdots(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_k(x) = \frac{\partial U_k(x)}{\partial x}$
$w(x,t) = \frac{\partial^{r+s} u(x,t)}{\partial x^r \partial t^s}$	$\frac{(k+s)!}{k!}\frac{\partial^r U_{k+s}(x)}{\partial t^r}$

3. Numerical Applications

3.1. **Problem 1.** Consider a two-dimensional incompressible Navier–Stokes equation [14, 18]

$$D_t u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g, \tag{3.1a}$$

$$D_t v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g, \tag{3.1b}$$

for surface flows or in the case of air or a perfect gas, we consider $g = 0$. The equations can be written as follows

$$D_t u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.2a}$$

$$D_t v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3.2b}$$

with the initial conditions

$$\begin{cases} u(x,y,0) = -\sin(x+y) \\ v(x,y,0) = \sin(x+y) \end{cases}. \tag{3.3}$$

In equations (3.2a) (3.2b), ρ_0 denotes the kinematic viscosity of the flow. ρ_0 is the ratio $\frac{\eta}{\rho}$, where η denotes dynamic viscosity of flow, and ρ the density of flow.

The application of RDTM to equations (3.2a) and (3.2b) gives the algorithm

$$\begin{aligned} &(k+1)\,V_{k+1}(x,y) + \sum_{r=0}^{k} U_r(x,y)\frac{\partial V_{k-r}(x,y)}{\partial x} + \sum_{r=0}^{k} V_r(x,y)\frac{\partial V_{k-r}(x,y)}{\partial y} \\ &= \rho_0\left(\frac{\partial^2 V_k}{\partial x^2} + \frac{\partial^2 V_k}{\partial y^2}\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &(k+1)\,U_{k+1}(x,y) + \sum_{r=0}^{k} U_r(x,y)\frac{\partial U_{k-r}(x,y)}{\partial x} + \sum_{r=0}^{k} V_r(x,y)\frac{\partial U_{k-r}(x,y)}{\partial y} \\ &= \rho_0\left(\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2}\right). \end{aligned} \tag{3.5}$$

The algorithm is manipulated by varying k, thus for $k = 0$:

$$(3.6)\qquad \begin{aligned} &V_1(x,y) + U_0(x,y)\frac{\partial V_0(x,y)}{\partial x} + V_0(x,y)\frac{\partial V_0(x,y)}{\partial y} \\ &= \rho_0 \left(\frac{\partial^2 V_0(x,y)}{\partial x^2} + \frac{\partial^2 V_0(x,y)}{\partial y^2} \right), \end{aligned}$$

$$(3.7)\qquad \begin{aligned} &U_1(x,y) + U_0(x,y)\frac{\partial U_0(x,y)}{\partial x} + V_0(x,y)\frac{\partial U_0(x,y)}{\partial y} \\ &= \rho_0 \left(\frac{\partial^2 U_0(x,y)}{\partial x^2} + \frac{\partial^2 U_0(x,y)}{\partial y^2} \right). \end{aligned}$$

Either

$$(3.8)\qquad \begin{aligned} &V_1(x,y) - \sin(x+y)\cos(x+y) + \sin(x+y)\cos(x+y) \\ &= \rho_0 \left(-\sin(x+y) - \sin(x+y) \right), \end{aligned}$$

$$(3.9)\qquad \begin{aligned} &U_1(x,y) + \sin(x+y)\cos(x+y) - \sin(x+y)\cos(x+y) \\ &= \rho_0 \left(\sin(x+y) + \sin(x+y) \right) \end{aligned}$$

$$(3.10)\qquad V_1 = -2\rho_0 \sin(x+y)$$

$$(3.11)\qquad U_1 = 2\rho_0 \sin(x+y)$$

For $k = 1$:

$$(3.12)\qquad \begin{aligned} &2V_2(x,y) + U_0\frac{\partial V_1}{\partial x} + U_1\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_1}{\partial x} + V_1\frac{\partial V_0}{\partial x} \\ &= \rho_0 \left(\frac{\partial^2 V_1(x,y)}{\partial x^2} + \frac{\partial^2 V_1(x,y)}{\partial y^2} \right), \end{aligned}$$

$$(3.13)\qquad \begin{aligned} &2U_2(x,y) + U_0\frac{\partial U_1}{\partial x} + U_1\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_1}{\partial x} + V_1\frac{\partial U_0}{\partial x} \\ &= \rho_0 \left(\frac{\partial^2 U_1(x,y)}{\partial x^2} + \frac{\partial^2 U_1(x,y)}{\partial y^2} \right). \end{aligned}$$

Either

$$(3.14a)\qquad V_2(x,y) = 2\rho_0^2 \sin(x+y),$$

(3.14b) $$U_2(x,y) = -2\rho_0^2 \sin(x+y).$$

By the same principle of iterations, the following expressions can be deduced

(3.15) $$V_3(x,y) = -\frac{4}{3}\rho_0^3 \sin(x+y), \qquad U_3(x,y) = +\frac{4}{3}\rho_0^3 \sin(x+y),$$

(3.16) $$V_4(x,y) = \frac{2}{3}\rho_0^4 \sin(x+y), \qquad U_4(x,y) = -\frac{2}{3}\rho_0^3 \sin(x+y).$$

Exploiting expressions (2.1) and (2.5) gives the solution in the form of a series. Let:

(3.17) $$\begin{aligned} v(x,y,t) = \sum_{k=0}^{\infty} V_k(x,y)t^k &= V_0(x,y) + V_1(x,y)t + V_2(x,y)t^2 \\ &+ V_3(x,y)t^3 + V_4(x,y)t^4 + \cdots , \end{aligned}$$

(3.18) $$\begin{aligned} u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k &= U_0(x,y) + U_1(x,y)t + U_2(x,y)t^2 \\ &+ U_3(x,y)t^3 + U_4(x,y)t^4 + \cdots , \end{aligned}$$

(3.19) $$\begin{aligned} v(x,y,t) &= \sin(x+y) - 2\rho_0 \sin(x+y)t + 2\rho_0^2 \sin(x+y)t^2 \\ &- \frac{4}{3}\rho_0^3 \sin(x+y)t^3 + \frac{2}{3}\rho_0^4 \sin(x+y)t^4 + \cdots , \end{aligned}$$

(3.20) $$\begin{aligned} u(x,y,t) &= -\sin(x+y) + 2\rho_0 \sin(x+y)t - 2\rho_0^2 \sin(x+y)t^2 \\ &+ \frac{4}{3}\rho_0^3 \sin(x+y)t^3 - \frac{2}{3}\rho_0^4 \sin(x+y)t^4 + \cdots , \end{aligned}$$

(3.21) $$v(x,y,t) = \sin(x+y)\left[1 - 2\rho_0 t + 2\rho_0^2 t^2 - \frac{4}{3}\rho_0^3 t^3 + \frac{2}{3}\rho_0^4 t^4 + \cdots\right],$$

(3.22) $$u(x,y,t) = -\sin(x+y)\left[1 - 2\rho_0 t + 2\rho_0^2 t^2 - \frac{4}{3}\rho_0^3 t^3 + \frac{2}{3}\rho_0^4 t^4 + \cdots\right],$$

$$v(x,y,t) = \sin(x+y)\left[1 + (-2\rho_0 t) + \frac{1}{2!}(-2\rho_0 t)^2 + \frac{1}{3!}(-2\rho_0 t)^3 + \frac{1}{4!}(-2\rho_0 t)^4 + \cdots\right], \tag{3.23}$$

$$u(x,y,t) = -\sin(x+y)\left[1 + (-2\rho_0 t) + \frac{1}{2!}(-2\rho_0 t)^2 + \frac{1}{3!}(-2\rho_0 t)^3 + \frac{1}{4!}(-2\rho_0 t)^4 + \cdots\right]. \tag{3.24}$$

The exact solution is:

$$v(x,y,t) = (\sin(x+y))\, exp\,(-2\rho_0 t) \tag{3.25a}$$

$$u(x,y,t) = -\,(\sin(x+y))\, exp\,(-2\rho_0 t) \tag{3.25b}$$

3.2. **Problem 2.**

The following nonlinear PDE systems are derived from fluid flow problems, turbulence, perturbation and many other phenomena [6]. The general case is as follows

$$\begin{cases} v_t(x,t) = u_{xx}(x,t) + \alpha u(x,t)u_x(x,t) - \beta\,(uv)_x \\ u_t(x,t) = v_{xx}(x,t) + \alpha v(x,t)v_x(x,t) - \beta\,(uv)_x \\ u(x,0) = f(x) \\ v(x,0) = g(x) \end{cases}. \tag{3.26}$$

This case takes into account the following values

$$\alpha = 2, \qquad \beta = \gamma = 1, \qquad f(x) = g(x) = \sin(x).$$

Substituting the above values gives the following (3.28) problem:

$$\begin{cases} v_t(x,t) = u_{xx}(x,t) + 2u(x,t)u_x(x,t) - (uv)_x \\ u_t(x,t) = v_{xx}(x,t) + 2v(x,t)v_x(x,t) - (uv)_x \end{cases}, \tag{3.27}$$

with the initial conditions

$$\begin{cases} u(x,0) = \sin(x) \\ v(x,0) = \sin(x) \end{cases}. \tag{3.28}$$

By analogy, the application of the RDTM to equations (3.28) gives the algorithm

$$\begin{cases} (k+1)\,U_{k+1} & = \dfrac{\partial^2 U_k}{\partial x^2} + 2\sum_{r=0}^{k} U_r \dfrac{\partial U_{k-r}}{\partial x} - \sum_{r=0}^{k} V_r \dfrac{\partial U_{k-r}}{\partial x} \\ & - \sum_{r=0}^{k} U_r \dfrac{\partial V_{k-r}}{\partial x} \\ (k+1)\,V_{k+1} & = \dfrac{\partial^2 V_k}{\partial x^2} + 2\sum_{r=0}^{k} V_r \dfrac{\partial V_{k-r}}{\partial x} - \sum_{r=0}^{k} V_r \dfrac{\partial U_{k-r}}{\partial x} \\ & - \sum_{r=0}^{k} U_r \dfrac{\partial V_{k-r}}{\partial x} \end{cases} \tag{3.29}$$

The algorithm is manipulated by varying k, thus for $k = 0$:

$$\begin{cases} U_1 = \dfrac{\partial^2 U_0}{\partial x^2} + 2U_0 \dfrac{\partial U_0}{\partial x} - V_0 \dfrac{\partial U_0}{\partial x} - U_0 \dfrac{\partial V_0}{\partial x} \\ V_1 = \dfrac{\partial^2 V_0}{\partial x^2} + 2V_0 \dfrac{\partial V_0}{\partial x} - V_0 \dfrac{\partial U_0}{\partial x} - U_0 \dfrac{\partial V_0}{\partial x} \end{cases} \tag{3.30}$$

Either

$$\begin{cases} U_1(x) = -\sin(x) + 2\sin(x)\cos(x) - \sin(x)\cos(x) - \sin(x)\cos(x) \\ V_1(x) = -\sin(x) + 2\sin(x)\cos(x) - \sin(x)\cos(x) - \sin(x)\cos(x) \end{cases} \tag{3.31}$$

So, after all calculations,

$$\begin{cases} U_1(x) = -\sin(x) \\ V_1(x) = -\sin(x) \end{cases}. \tag{3.32}$$

For $k = 1$:

$$\begin{cases} 2U_2 & = \dfrac{\partial^2 U_1}{\partial x^2} + 2U_0 \dfrac{\partial U_1}{\partial x} + 2U_1 \dfrac{\partial U_0}{\partial x} - V_0 \dfrac{\partial U_1}{\partial x} \\ & -V_1 \dfrac{\partial U_0}{\partial x} - U_0 \dfrac{\partial V_1}{\partial x} - U_1 \dfrac{\partial V_0}{\partial x} \\ 2V_2 & = \dfrac{\partial^2 V_1}{\partial x^2} + 2V_0 \dfrac{\partial V_1}{\partial x} + 2V_1 \dfrac{\partial V_0}{\partial x} - V_0 \dfrac{\partial U_1}{\partial x} \\ & -V_1 \dfrac{\partial U_0}{\partial x} - U_0 \dfrac{\partial V_1}{\partial x} - U_1 \dfrac{\partial V_0}{\partial x} \end{cases}. \tag{3.33}$$

Either

$$U_2(x,y) = \frac{1}{2}\sin(x), \qquad V_2(x,y) = \frac{1}{2}\sin(x). \tag{3.34}$$

For $k = 2$:

$$\begin{cases} 3U_3 & = \dfrac{\partial^2 U_2}{\partial x^2} + 2\left(U_0\dfrac{\partial U_2}{\partial x} + U_1\dfrac{\partial U_1}{\partial x} + U_2\dfrac{\partial U_0}{\partial x}\right) - V_0\dfrac{\partial U_2}{\partial x} - V_1\dfrac{\partial U_1}{\partial x} \\ & -V_2\dfrac{\partial U_0}{\partial x} - U_2\dfrac{\partial V_0}{\partial x} - U_0\dfrac{\partial V_2}{\partial x} - U_1\dfrac{\partial V_1}{\partial x} \\ 3V_3 & = \dfrac{\partial^2 V_2}{\partial x^2} + 2\left(V_0\dfrac{\partial V_2}{\partial x} + V_1\dfrac{\partial V_1}{\partial x} + V_2\dfrac{\partial V_0}{\partial x}\right) - V_0\dfrac{\partial U_2}{\partial x} - V_1\dfrac{\partial U_1}{\partial x} \\ & -V_2\dfrac{\partial U_0}{\partial x} - U_2\dfrac{\partial V_0}{\partial x} - U_0\dfrac{\partial V_2}{\partial x} - U_1\dfrac{\partial V_1}{\partial x} \end{cases}. \tag{3.35}$$

Either after all calculations

$$U_3(x) = -\frac{1}{6}\sin(x), \qquad V_3(x) = -\frac{1}{6}\sin(x). \tag{3.36}$$

By the same principle of iterations, the following expressions can be deduced

$$U_4(x) = \frac{1}{24}\sin(x), \qquad V_4(x) = \frac{1}{24}\sin(x), \tag{3.37}$$

$$U_5(x) = -\frac{1}{120}\sin(x), \qquad V_5(x) - = \frac{1}{120}\sin(x). \tag{3.38}$$

Little by little, the following expressions result.

Exploiting expressions (2.1) and (2.5) gives the solution in the form of a series. Let:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \qquad \sin(x)v(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k. \tag{3.39}$$

Writing as an extension of the expressions (3.39) gives:

$$\begin{aligned} u(x,t) = \sin(x) - \sin(x)t + \frac{1}{2}\sin(x)t^2 - \frac{1}{6}\sin(x)t^3 + \frac{1}{24}\sin(x)t^4 \\ - \frac{1}{120}\sin(x)t^5 + \cdots . \end{aligned}$$

Either

$$\begin{aligned} u(x,t) &= \sin(x)\left(-t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \cdots\right), \\ v(x,t) &= \sin(x)\left(-t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \cdots\right). \end{aligned}$$

The exact solutions are given by

$$u(x,t)=e^{-t}\sin(x), \qquad v(x,t)=e^{-t}\sin(x). \tag{3.40}$$

3.3. **Problem 3.**

Consider the following system of PDEs [2]

$$\begin{cases} u_t(x,y,t)=u_{xx}+u_{yy}+2u\left(u_x+u_y\right)-(uv)_x-(uv)_y \\ v_t(x,y,t)=v_{xx}+v_{yy}+2u\left(v_x+v_y\right)-(uv)_x-(uv)_y \\ u(x,y,0)=v(x,y,0)=\cos(x+y) \end{cases}. \tag{3.41}$$

Applying the RDTM to the problem (3.41) gives the following algorithm:

$$\begin{cases} (k+1)\,U_{k+1}(x,y)=\dfrac{\partial^2 U_k(x,y)}{\partial x^2}+\dfrac{\partial^2 U_k(x,y)}{\partial y^2} \\ +2\left(\displaystyle\sum_{r=0}^{k} U_r\frac{\partial U_{k-r}(x,y)}{\partial x}+\sum_{r=0}^{k} U_r\frac{\partial U_{k-r}(x,y)}{\partial y}\right) \\ -\displaystyle\sum_{r=0}^{k} V_r\frac{\partial U_{k-r}}{\partial x}-\sum_{r=0}^{k} U_r(x,y)\frac{\partial V_{k-r}}{\partial x}-\sum_{r=0}^{k} V_r\frac{\partial U_{k-r}}{\partial y}-\sum_{r=0}^{k} U_r\frac{\partial V_{k-r}}{\partial y} \\ (k+1)\,V_{k+1}(x,y)=\dfrac{\partial^2 V_k(x,y)}{\partial x^2}+\dfrac{\partial^2 V_k(x,y)}{\partial y^2} \\ +2\left(\displaystyle\sum_{r=0}^{k} U_r\frac{\partial V_{k-r}(x,y)}{\partial x}+\sum_{r=0}^{k} U_r\frac{\partial V_{k-r}(x,y)}{\partial y}\right) \\ -\displaystyle\sum_{r=0}^{k} V_r\frac{\partial U_{k-r}}{\partial x}-\sum_{r=0}^{k} U_r\frac{\partial V_{k-r}}{\partial x}-\sum_{r=0}^{k} V_r\frac{\partial U_{k-r}}{\partial y}-\sum_{r=0}^{k} U_r\frac{\partial V_{k-r}}{\partial y} \end{cases}. \tag{3.42}$$

For $k=0$:

$$\begin{cases} U_1(x,y) \\ =\dfrac{\partial^2 U_0(x,y)}{\partial x^2}+\dfrac{\partial^2 U_0(x,y)}{\partial y^2}+2\left(U_0\dfrac{\partial U_0(x,y)}{\partial x}+U_0\dfrac{\partial U_0(x,y)}{\partial y}\right) \\ -V_0\dfrac{\partial U_0}{\partial x}-U_0(x,y)\dfrac{\partial V_0}{\partial x}-V_0\dfrac{\partial U_0}{\partial y}-U_0\dfrac{\partial V_{k-r}}{\partial y} \\ V_1(x,y) \\ =\dfrac{\partial^2 V_0(x,y)}{\partial x^2}+\dfrac{\partial^2 V_0(x,y)}{\partial y^2}+2\left(U_0\dfrac{\partial V_0(x,y)}{\partial x}+U_0\dfrac{\partial V_0(x,y)}{\partial y}\right) \\ -V_0\dfrac{\partial U_0}{\partial x}-U_0\dfrac{\partial V_0}{\partial x}-V_0\dfrac{\partial U_0}{\partial y}-U_0\dfrac{\partial V_0}{\partial y} \end{cases}. \tag{3.43}$$

After all calculations have been made, the following results are obtained

$$U_1 = -2\cos(x+y), \qquad V_1 = -2\cos(x+y). \tag{3.44}$$

Applying the algorithm (3.42), $k = 2$, $k = 3$, $k = 4$, and step by step, the following results are obtained:

$$U_2 = 2\cos(x+y), \qquad V_2 = 2\cos(x+y), \tag{3.45}$$

$$U_3 = -\frac{4}{3}\cos(x+y), \qquad V_3 = -\frac{4}{3}\cos(x+y), \tag{3.46}$$

$$U_4 = \frac{2}{3}\cos(x+y), \qquad V_4 = \frac{2}{3}\cos(x+y), \tag{3.47}$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

Exploiting expressions (2.1) and (2.5) gives the solution in the form of a series. Let:

$$\begin{cases} u(x,y,t) & = \cos(x+y) - 2\cos(x+y)t + 2\cos(x+y)t^2 \\ & \quad -\frac{4}{3}\cos(x+y)t^3 + \frac{2}{3}\cos(x+y)t^4 + \cdots , \\ v(x,y,t) & = \cos(x+y) - 2\cos(x+y)t + 2\cos(x+y)t^2 \\ & \quad -\frac{4}{3}\cos(x+y)t^3 + \frac{2}{3}\cos(x+y)t^4 + \cdots . \end{cases} \tag{3.48}$$

Transformation of the expressions gives

$$\begin{cases} u(x,y,t) = \cos(x+y)\left[1 + (-2t) + \frac{(-t)^2}{2!} - \frac{(-t)^2}{3!} + \frac{(-t)^4}{4!} + \cdots\right] \\ v(x,y,t) = \cos(x+y)\left[1 + (-2t) + \frac{(-t)^2}{2!} - \frac{(-t)^2}{3!} + \frac{(-t)^4}{4!} + \cdots\right] \end{cases} . \tag{3.49}$$

The exact solution to the problem (3.41) is:

$$u(x,y,t) = \exp(-2t)\cos(x+y), \qquad v(x,y,t) = \exp(-2t)\cos(x+y). \tag{3.50}$$

4. Conclusion

Generally speaking, the search for exact solutions to the equations modeling flows, specifically the solutions to the Navier-Stokes equations, has not always been easy. Great difficulties have often existed. However, RDTM was applied with ease. Good results were obtained. The expected results are the exact solutions of the proposed problems. Exact solutions were obtained. Of course, the calculations are tedious and the method requires a thorough grasp of the basic concepts. The results obtained confirm the effectiveness of the method. It should be noted that It should be emphasized that finding the solution requires a good command of Taylor series manipulation. .

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