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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF SEQUENCES OF NEGATIVELY DEPENDENT RANDOM VARIABLES

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ABSTRACT. This paper is a theoretical contribution on the complete convergence of partial sums. Let $\{X_n, n \geq 1\}$ be a sequence of non negatively dependent random, which is stochastically dominated by a random variable X and $\{\ \Psi_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a an array of random variables. Under mild condition we establish the complete convergence for weighted sums $\sum_{i=1}^{j} \Psi_{ni} X_i$. This result obtained with random coefficients generalizes the work of those obtained with real coefficients [12–14, 16]. Our results also generalize those on complete convergence theorem previously obtained from the independent and identically distributed case to negatively dependent.

1. INTRODUCTION

In many stochastic models, the assumption that random variables are independent is not realistic, especially in the case of real data. It is therefore of interest to extend the results from independent framework to dependent variables. This work is part of this vast project to extend the results obtained with independent variables in the dependent case.

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Hence, extending the limit properties of independent or random variables to the case of negatively dependent (ND) variables is highly desirable and of considerable significance in the theory and application.

The concept of ND was introduced by Bozorgnia et al. [2], these concepts of dependent random variables have been useful in reliability theory and applications. The notion of ND has been the subject of many research works in recent years and very interesting results have been established (cf [2, 4-10]).

The main purpose of the paper is to prove the complete convergence for weighted sums of sequences of negatively dependent with random coefficients, this can be considered as a generalization of [14]. We can not that, this work is a continuation of the those we are carring out on the processes defined in abstract for this see [27].

Definition 1.1. *Random variables X and Y 0are said to be negatively dependent (ND) if*

$$P\left(X \leq x, Y \leq y\right) \leq P\left(X \leq x\right) P\left(Y \leq y\right), \tag{1.1}$$

for all $x, y \in \mathbb{R}$. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1)*. It is important to note that* (1.1) *implies:*

$$P\left(X > x, Y > y\right) \leq P\left(X > x\right) P\left(Y > y\right), \tag{1.2}$$

for all $x, y \in \mathbb{R}$. Moreover, it follows that (1.2) *implies* (1.1)*, and hence,* (1.2) *and* (1.1) *are equivalent. However,* (1.1) *and* (1.2) *are not equivalent for a collection of 3 or more random variables.Consequently, the following definition is needed to define sequences of negatively dependent random variables.*

Definition 1.2. *The random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if for all real $x_1, \ldots, x_n$,*

$$\mathbb{P}\left(\cap_{j=1}^{n} X_j \leq x_j\right) \leq \prod_{j=1}^{n} \mathbb{P}\left(X_j \leq x_j\right), \tag{1.3}$$

$$\mathbb{P}\left(\cap_{j=1}^{n} X_j > x_j\right) \leq \prod_{j=1}^{n} \mathbb{P}\left(X_j > x_j\right). \tag{1.4}$$

An infinite sequence of random variables $\{X_n, n \geq 1\}$ is said to be ND if every finite subset $X_1, \ldots, X_n$ is ND.

The concept of stochastic domination is as follows.

Definition 1.3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that $\mathbb{P}(|X_n| > x) \leq C\mathbb{P}(|X| > x)$ for all $x \geq 0$ and $n \geq 1$.*

Definition 1.4. *A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant θ (write $X_n \to \theta$ completely) if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - \theta| > \varepsilon) < \infty. \tag{1.5}$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence. Hsu and Robbins [7] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdos [8] proved the converse. The result of Hsu-Robbins-Erdos is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz [9] for the strong law of large numbers. Due to Baum and Katz [9], one has:

Theorem 1.1. *Suppose that $p > 1 < \frac{1}{\alpha}$ et $\frac{1}{2} < \alpha < 1$ Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables satisfying $\mathbb{E}X = 0$, which is stochastically dominated by a random variable X with $\mathbb{E}|X_1| = 0$. Then the following statements are equivalent:*

(1) $\mathbb{E}|X_1|^p < \infty$
(2) $\sum_{n=1}^{\infty} n^{p\alpha-2}\mathbb{P}\left(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| > \varepsilon n^{\alpha}\right) < \infty$ *for all* $\varepsilon > 0$.

In this paper, we study the complete convergence for negatively dependent random variables, with the hypothesis $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ random, so we can consider our results as a generalization of the previous Theorem.

2. Main result and proof

To establish the main result for this work, we will need to assume the following:

$\mathbf{H_1}-$ We suppose $(\Psi_{i1}, \Psi_{i2}, \ldots)$ is independent of X_i for any $i \geq 1$ and

$$\sum_{j\geq 1} \mathbb{E}|\Psi_{ij}| < \infty \quad \text{and} \quad \sum_{j\geq 1} \mathbb{E}|\Psi_{ij}|^2 < \infty. \tag{2.1}$$

$\mathbf{H_2}-$ We assume that the random $\Psi_{ik}(t)$ has an upper endpoint c_k defined by

$$c_k = sup\{c : \ \mathbb{P}(|\Psi_{ik}(t)| \leq c) < 1\}, \quad k = 1, 2, \ldots. \tag{2.2}$$

and there exists $\delta > 0$ such that $\sum_{k=1}^{\infty} c_k^{1-\delta} < 1$, $\sum_{k=1}^{\infty} c_k^{\alpha\delta} < \infty$.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables, which is stochastically dominated by a random variable X with $\mathbb{E}|X|^\beta < \infty$ for some $\beta > \frac{1}{\alpha}$ and $\frac{1}{2} \leq \alpha \leq 1$. Let $\{\ \Psi_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a sequence of random variables independent of $\{X_n, n \geq 1\}$ and satisfying $\mathbf{H_1}$ and $\mathbf{H_2}$. Suppose that 1.3 holds for $\{X_n, n \geq 1\}$. Then*

$$\sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(\max_{1\leq j\leq n} |\sum_{i=1}^{j} \Psi_{ni}X_i| > \varepsilon n^\alpha\right) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{2.3}$$

To prove the main results of the paper, we need the following useful lemma. For the proof and more datails, one can refer to Wu [14, 23].

Lemma 2.1. *(See [2]) Let $X_1, \ldots, X_n$ be ND random variables and let $\{f_n, n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing), then $\{f_n(X_n), n \geq 1\}$ is still a sequence of ND r.v.s.*

Lemma 2.2. *(See [20]) Let $\{X_n, n \geq 1\}$ be an ND sequence with $\mathbb{E}X_n = 0$ and $\mathbb{E}|X_n|^p < \infty$ for $p \geq 2$, then*

$$\mathbb{E}|S_n|^p \leq c_p \left\{\sum_{i=1}^{n} \mathbb{E}|X_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}X_i^2\right)^{\frac{p}{2}}\right\}, \tag{2.4}$$

where $c_p > 0$ depends only on p.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$\mathbb{E}|X_n|^\alpha \mathbb{I}_{|X_n|\le b} \le C_1\left[\mathbb{E}|X|^\alpha \not{1}_{|X|\le b} + b^\alpha \mathbb{P}\left(|X|>b\right)\right] \tag{2.5}$$

and

$$\mathbb{E}|X_n|^\alpha \mathbb{I}_{|X_n|>b} \le C_2 \mathbb{E}|X|^\alpha \mathbb{I}_{|X|>b}, \tag{2.6}$$

where C_1 and C_2 are positive constants. Consequently, $\mathbb{E}|X_n|^\alpha \le C\mathbb{E}|X|^\alpha$.

Proof. of Theorem 2.1 For $1 \le i \le n$ and $n \ge 1$, define $X'_{ni} = X_i \mathbb{I}_{|X_i|\le n^\alpha}$. Using $\mathbf{H_1}$ and $\mathbf{H_2}$, we have that:

$$\begin{aligned}
n^{-\alpha}\max_{1\le j\le n}|\sum_{i=1}^{j}\mathbb{E}(\Psi_{ni}X'_{ni})| &= n^{-\alpha}\max_{1\le j\le n}|\sum_{i=1}^{j}\mathbb{E}(\Psi_{ni})\mathbb{E}(X_i\mathbb{I}_{|X_i|>n^\alpha})| \\
&\le n^{-\alpha}\max_{1\le j\le n}\sum_{i=1}^{j}|\mathbb{E}(\Psi_{ni})\mathbb{E}X_i\mathbb{I}_{|X_i|>n^\alpha}| \\
&\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}(\Psi_{ni})|\mathbb{E}|X_i|\mathbb{I}_{|X_i|>n^\alpha} \\
&\le n^{-\alpha}\sum_{i=1}^{n}\mathbb{E}|\Psi_{ni}|\mathbb{E}|X_i|\mathbb{I}_{|X_i|>n^\alpha} \\
&\le n^{-\alpha}\sum_{i=1}^{n}\mathbb{E}|\Psi_{ni}|\mathbb{E}|X_i|\mathbb{I}_{|X_i|>n^\alpha} \\
&\le n^{-\alpha}\sum_{i=1}^{\infty}\mathbb{E}|\Psi_{ni}|\mathbb{E}|X_i|\mathbb{I}_{|X_i|>n^\alpha} \\
&\le Cn^{1-\alpha}\mathbb{E}|X|\mathbb{I}_{|X|>n^\alpha}\sum_{i=1}^{\infty}\mathbb{E}|\Psi_{ni}| \\
&\le Cn^{1-\frac{1}{\beta}}\mathbb{E}|X|\mathbb{I}_{|X|>c^{-1}n^{\frac{1}{\beta}}}\sum_{i=1}^{\infty}\mathbb{E}|\Psi_{ni}| \\
&\le Cn^{1-\frac{1}{\beta}}\mathbb{E}|X|^\beta\sum_{i=1}^{\infty}\mathbb{E}|\Psi_{ni}| \\
&\to 0,\ as\ n\to\infty
\end{aligned}$$

From that we have for all n large enough $n^{-\alpha}\max_{1\le j\le n}|\sum_{i=1}^{j}\mathbb{E}\left(a_{ni}X'_{ni}\right)| < \frac{\varepsilon}{2}$. It follows that:

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(\max_{1\leq j\leq n}|\sum_{i=1}^{j}\Psi_{ni}X_i|>\varepsilon n^{\alpha}\right)\\
\leq & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{n}\Psi_{ni}X_i|>\varepsilon n^{\alpha}\right)\\
= & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{n}\Psi_{ni}X_i-\sum_{i=1}^{j}\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)+\sum_{i=1}^{j}\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)|>\varepsilon n^{\alpha}\right)\\
\leq & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{n}\Psi_{ni}X_i-\sum_{i=1}^{j}\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)|>\varepsilon n^{\alpha}\right)\\
+ & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{j}\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)|>\varepsilon n^{\alpha}\right)\\
\leq & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{n}\left(\Psi_{ni}X_i-\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)\right)|>\varepsilon n^{\alpha}\right)\\
+ & \sum_{n=1}^{\infty} n^{\beta\alpha-2}\mathbb{P}\left(|\sum_{i=1}^{n}\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)|>\varepsilon n^{\alpha}\right)\\
\leq & R+T.
\end{aligned}$$

By second Markov inegality we have: $R \leq \sum_{n=1}^{\infty} n^{\beta\alpha-2}\frac{\mathbb{E}|\sum_{i=1}^{n}\left(\Psi_{ni}X_i-\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)\right)|^2}{\varepsilon^2 n^{2\alpha}}$
Applying Lemma 2.2, $\mathbf{H_1}$ and $\mathbf{1.3}$ we have:

$$\begin{aligned}
R \leq & \sum_{n=1}^{\infty}\varepsilon^{-2}n^{\alpha(\beta-2)-2}c_2\left\{\sum_{i=1}^{n}\mathbb{E}|\left(\Psi X_i-\mathbb{E}\left(\Psi X'_{ni}\right)\right)|^2\right.\\
+ & \left.\left(\sum_{i=1}^{n}\mathbb{E}\left[\left(\Psi_{ni}X_i-\mathbb{E}\left(\Psi X'_{ni}\right)\right)\right]^2\right)^{\frac{2}{2}}\right\}\\
\leq & \sum_{n=1}^{\infty}\varepsilon^{-2}n^{\alpha(\beta-2)-2}c_2\left\{\sum_{i=1}^{n}\mathbb{E}|\left(\Psi_{ni}X_i-\mathbb{E}\left(\Psi X'_{ni}\right)\right)|^2\right.\\
+ & \left.\sum_{i=1}^{n}\mathbb{E}|\left(\Psi X_i-\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)\right)|^2\right\}\\
\leq & \sum_{n=1}^{\infty}\varepsilon^{-2}n^{\alpha(\beta-2)-2}2c_2\sum_{i=1}^{n}\mathbb{E}|\left(\Psi_{ni}X_i-\mathbb{E}\left(\Psi X'_{ni}\right)\right)|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-2} 2c_2 \sum_{i=1}^{n} \left[\mathbb{E}\left((\Psi_{ni}X_i)^2\right) - \left(\mathbb{E}\left(\Psi_{ni}X'_{ni}\right)\right)^2\right] \\
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-2} 2c_2 \sum_{i=1}^{n} \left[\mathbb{E}\left(\Psi_{ni}X_i\right)^2 - \mathbb{E}|\Psi_{ni}|^2 \left(\mathbb{E}X'_{ni}\right)^2\right] \\
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-2} 2c_2 \sum_{i=1}^{n} \left[\mathbb{E}|\Psi_{ni}|^2 \mathbb{E}\left(X_i\right)^2 - \mathbb{E}|\Psi_{ni}|^2 \left(\mathbb{E}X'_{ni}\right)^2\right] \\
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-2} 2c_2 \sum_{i=1}^{n} \mathbb{E}|\Psi_{ni}|^2 \left[\mathbb{E}|X_i|^2 - \left(\mathbb{E}X_i \mathbb{1}_{|X_i|\le n^{\alpha}}\right)^2\right] \\
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-2} 2c_2 \sum_{i=1}^{\infty} \mathbb{E}|\Psi_{ni}|^2 \sum_{i=1}^{n} \left[\mathbb{E}|X_i|^2 - \left(\mathbb{E}X_i\right)^2\right] \\
&\le \sum_{n=1}^{\infty} \varepsilon^{-2} n^{\alpha(\beta-2)-1} 2c_2 \sum_{i=1}^{\infty} \mathbb{E}|\Psi_{ni}|^2 \sum_{i=1}^{n} \mathbb{V}\mathrm{ar}(X_i) \\
&< \infty
\end{aligned}
$$

Using the Markov inegality, $\mathbf{H_4}$ and $\mathbf{2.12}$ we have:

$$
\begin{aligned}
T &= \sum_{n=1}^{\infty} n^{\beta\alpha-2} \mathbb{P}\left(\left|\sum_{i=1}^{n} \mathbb{E}\left(\Psi_{ni}X'_{ni}\right)\right| > \varepsilon n^{\alpha}\right) \\
&\le C \sum_{n=1}^{\infty} n^{\beta\alpha-2} \mathbb{P}\left(\left|\sum_{i=1}^{n} \Psi_{ni}\left(X'_{ni} - \mathbb{E}\left(X'_{ni}\right)\right)\right| > \varepsilon n^{\alpha}\right) \\
&\le C \sum_{n=1}^{\infty} n^{\beta\alpha-2} \mathbb{P}\left(\sum_{i=1}^{n} |\Psi_{ni}|\left|\left(X'_{ni} - \mathbb{E}\left(X'_{ni}\right)\right)\right| > \varepsilon n^{\alpha}\right) \\
&\le C \sum_{n=1}^{\infty} n^{\beta\alpha-2} \mathbb{P}\left(\sum_{i=1}^{n} c_i\left|X'_{ni} - \mathbb{E}\left(X'_{ni}\right)\right| > \varepsilon n^{\alpha}\right) \\
&\le C \left(\frac{2}{\varepsilon}\right)^{r} \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \mathbb{E}\left(\sum_{i=1}^{n} c_i\left|X'_{ni} - \mathbb{E}\left(X'_{ni}\right)\right|\right)^{r} \\
&\le C \left(\frac{2}{\varepsilon}\right)^{r} \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \left[\sum_{i=1}^{n} c_i^r \mathbb{E}|X'_{ni} - \mathbb{E}X'_{ni}|^r\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\sum_{i=1}^{n} c_i^2 \mathbb{E}\left(X'_{ni} - \mathbb{E}X'_{ni}\right)^2\right)^{\frac{r}{2}}\Bigg] \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{n} c_i^r \mathbb{E}|X'_{ni}|^r \\
&+ C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \left(\sum_{i=1}^{n} c_i^2 \mathbb{E}\left(X'_{ni}\right)^2\right)^{\frac{r}{2}} \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{i=1}^{n} \mathbb{E}|X_i|^r \mathbb{1}_{|X_i|\leq n^\alpha} \\
&+ C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{\infty} c_i^{\alpha r} \left(\sum_{i=1}^{n} \mathbb{E}\left(X_i^2 \mathbb{1}_{|X_i|\leq n^\alpha}\right)\right)^{\frac{r}{2}} \\
&\leq CT_1 + CT_2
\end{aligned}
$$

Suppose $r > \beta$. Applying $Lemma\mathbf{2.3}$ we have:

$$
\begin{aligned}
T_1 &\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{i=1}^{n} \left[\mathbb{E}|X|^r \mathbb{1}_{|X|\leq n^\alpha} + n^{\alpha r}\mathbb{P}\left(|X| > n^\alpha\right)\right] \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-1} \mathbb{E}|X|^r \mathbb{1}_{|X|\leq n^\alpha} \\
&+ C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{n=1}^{\infty} n^{\alpha\beta-1} \mathbb{P}\left(|X| > n^\alpha\right) \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-1} \sum_{i=1}^{n} \mathbb{E}|X|^r \mathbb{1}_{(i-1)^\alpha<|X|\leq i^\alpha} + C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \sum_{i=1}^{\infty} \mathbb{E}|X|^r \mathbb{1}_{(i-1)^\alpha<|X|\leq i^\alpha} \sum_{n=i}^{\infty} n^{\alpha(\beta-r)-1} + C \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \left(\sum_{i=1}^{\infty} \mathbb{E}|X|^r \mathbb{1}_{(i-1)^\alpha<|X|\leq i^\alpha} i^{\alpha(\beta-r)} + C\right) \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{r\alpha} \left(\sum_{i=1}^{\infty} i^{\alpha(\beta-r)} \mathbb{E}|X|^\beta + C\right) \\
&< \infty
\end{aligned}
$$

Taking $\beta \geq 2$, $r > \max\left\{\frac{\alpha\beta-1}{\alpha-\frac{1}{2}}, \beta\right\}$ and applying $Lemma\mathbf{2.3}$, we obtain:

$$\begin{aligned}
T_2 &\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{\infty} c_i^{\alpha r}\left(\sum_{i=1}^{n}\left[\mathbb{E}|X|^2\mathbb{1}_{|X|\leq n^\alpha} + n^{2\alpha}\mathbb{P}\left(|X|>n^\alpha\right)\right]\right)^{\frac{r}{2}} \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2} \sum_{i=1}^{\infty} c_i^{\alpha r}\left(\sum_{i=1}^{n}\left[\mathbb{E}|X|^2\mathbb{1}_{|X|\leq n^\alpha} + \mathbb{E}|X|^2\mathbb{1}_{|X|\leq n^\alpha}\right]\right)^{\frac{r}{2}} \\
&\leq C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{\alpha r} \sum_{n=1}^{\infty} n^{\alpha(\beta-r)-2+\frac{r}{2}} \\
&= C\left(\frac{2}{\varepsilon}\right)^r \sum_{i=1}^{\infty} c_i^{\alpha\delta r} \sum_{n=1}^{\infty} n^{\alpha\beta-2-r(\alpha-\frac{1}{2})} \\
&< \infty
\end{aligned}$$

This ends the proof. □

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