J COMP SCI
APPL MATH

Journal of Computer Science and Applied Mathematics
Vol. 7, no.2, (2025), 35–40
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.7.2.1

ON A WEIGHTED RATIO-TYPE INTEGRAL INEQUALITY

Christophe Chesneau

ABSTRACT. This note presents a new weighted ratio-type integral inequality, accompanied by a comprehensive proof. In addition, we derive another integral inequality of a related form.

1. INTRODUCTION

The scope of general integral inequalities is vast. Various types serve distinct purposes across fields such as mathematical analysis, probability theory, information theory, and the applied sciences. Hence, the development of new and refined classes of inequalities remains an active area of research. For the purposes of this note, we focus on two ratio-type integral inequalities introduced and studied in [1, 2]. The inequality [1, Theorem 7] is stated below. Let $a, b \in \mathbb{R}$ with $b > a$, $f, g, h : [a, b] \to [0, +\infty)$ be three functions such that, for any $x, y \in [a, b]$, we have

$$(g(x) - g(y)) \left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)} \right) \geq 0.$$

Key words and phrases. integral inequalities, weight functions, ratio functions.
Submitted: 29.11.2025; *Accepted:* 15.12.2025; *Published:* 02.01.2026.

Then the following inequality holds:

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\int_a^b f(x)g(x)dx}{\int_a^b h(x)g(x)dx}.$$

On this foundation, [1] introduced several additional ratio-type integral inequalities, which were subsequently refined in [3]. A distinct ratio-type integral inequality was later established in [2], as stated below. Let $a, b \in \mathbb{R}$ with $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be two functions such that f/g and fg are both non-decreasing. Then the following inequality holds:

$$\frac{\int_a^b 1/f(x)dx}{\int_a^b 1/g(x)dx} \geq \frac{\int_a^b g(x)dx}{\int_a^b f(x)dx}.$$

This inequality is of particular interest because it establishes a reciprocal relationship between the integrals of f and g, linking their harmonic and arithmetic means under monotonicity conditions. It has potential applications in analysis and optimization.

In this note, we generalize the above inequality by introducing a weight function, which provides an additional degree of flexibility. We then utilize this framework to establish a new type of integral inequality.

The main results are presented in the next section, followed by concluding remarks in Section 3.

2. Main results

Our new weighted ratio-type integral inequality is formally stated in the theorem below.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, $f, g : [a, b] \to [0, +\infty)$ be two functions such that f/g and fg are both non-decreasing, and $w : [a, b] \to [0, +\infty)$ be a (weight) function. Then the following inequality holds:*

$$\frac{\int_a^b w(x)/f(x)dx}{\int_a^b w(x)/g(x)dx} \geq \frac{\int_a^b g(x)w(x)dx}{\int_a^b f(x)w(x)dx}.$$

Proof. The proof follows the approach in [2], adapted to incorporate the weight function w. For any $x \in [a,b]$, let us consider

$$\Omega(x) = \left(\int_a^x f(t)w(t)dt\right)\left(\int_a^x \frac{w(t)}{f(t)}dt\right) - \left(\int_a^x g(t)w(t)dt\right)\left(\int_a^x \frac{w(t)}{g(t)}dt\right).$$

Using standard derivative rules and appropriate developments, we find that

$$\begin{aligned}\Omega'(x) &= f(x)w(x)\left(\int_a^x \frac{w(t)}{f(t)}dt\right) + \left(\int_a^x f(t)w(t)dt\right)\frac{w(x)}{f(x)} \\ &- g(x)w(x)\left(\int_a^x \frac{w(t)}{g(t)}dt\right) - \left(\int_a^x g(t)w(t)dt\right)\frac{w(x)}{g(x)} \\ &= w(x)\int_a^x \left(\frac{f(x)}{f(t)} + \frac{f(t)}{f(x)} - \frac{g(x)}{g(t)} - \frac{g(t)}{g(x)}\right) w(t)dt \\ &= w(x)\int_a^x \frac{(f(x)g(t) - f(t)g(x))(f(x)g(x) - f(t)g(t))}{f(x)f(t)g(x)g(t)} w(t)dt \\ &= \frac{w(x)}{f(x)}\int_a^x \left(\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}\right)(f(x)g(x) - f(t)g(t))\frac{w(t)}{f(t)}dt.\end{aligned}$$

Since f/g and fg are both non-decreasing, for any $t \in [a,x]$, we have

$$\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \geq 0, \quad f(x)g(x) - f(t)g(t) \geq 0.$$

This combined with the facts that f and w are non-negative implies that $\Omega'(x) \geq 0$. Therefore, Ω is non-decreasing, and, for any $x \in [a,b]$, we have $\Omega(x) \geq \Omega(a) = 0$, i.e.,

$$\left(\int_a^x f(t)w(t)dt\right)\left(\int_a^x \frac{w(t)}{f(t)}dt\right) - \left(\int_a^x g(t)w(t)dt\right)\left(\int_a^x \frac{w(t)}{g(t)}dt\right) \geq 0.$$

Since the integrals involved are non-negative, this is equivalent to

$$\frac{\int_a^x w(t)/f(t)dt}{\int_a^x w(t)/g(t)dt} \geq \frac{\int_a^x g(t)w(t)dt}{\int_a^x f(t)w(t)dt}.$$

By taking $x = b$ and rearranging a bit the notation, we get

$$\frac{\int_a^b w(x)/f(x)dx}{\int_a^b w(x)/g(x)dx} \geq \frac{\int_a^b g(x)w(x)dx}{\int_a^b f(x)w(x)dx}.$$

This completes the proof of the theorem. □

By taking $w = 1$, Theorem 2.1 reduces to the result obtained in [2]. If $a \geq 0$, the choice of the power weight function $w(x) = x^\alpha$ with $\alpha \in \mathbb{R}$ leads to

$$\frac{\int_a^b x^\alpha / f(x)dx}{\int_a^b x^\alpha / g(x)dx} \geq \frac{\int_a^b g(x)x^\alpha dx}{\int_a^b f(x)x^\alpha dx}.$$

It is worth noting that Theorem 2.1 remains valid if f/g and fg are both non-increasing rather than non-decreasing. Conversely, if the two functions exhibit opposite monotonicity, the direction of the inequality is reversed. If we analyze the proof of Theorem 2.1, it is in fact proved that, for any $x \in [a, b]$, we have

$$\frac{\int_a^x w(t)/f(t)dt}{\int_a^x w(t)/g(t)dt} \geq \frac{\int_a^x g(t)w(t)dt}{\int_a^x f(t)w(t)dt},$$

which is a bit more general.

The absence of any restriction on the weight function w is a clear advantage of Theorem 2.1, as it allows for greater flexibility and adaptability to various mathematical settings. To illustrate this point, we present a new theorem whose proof is derived from Theorem 2.1.

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be two functions such that f/g and fg are both non-decreasing. Then the following inequalities hold:*

$$\frac{(b-a)\int_a^b g^2(x)dx}{\int_a^b g(x)/f(x)dx} \leq \int_a^b f(x)g(x)dx \leq \frac{(b-a)\int_a^b f^2(x)dx}{\int_a^b f(x)/g(x)dx},$$

Proof. Applying Theorem 2.1 with $w = f$ yields

$$\frac{b-a}{\int_a^b f(x)/g(x)dx} \geq \frac{\int_a^b g(x)f(x)dx}{\int_a^b f^2(x)dx},$$

giving the following inequality:

$$\int_a^b f(x)g(x)dx \leq \frac{(b-a)\int_a^b f^2(x)dx}{\int_a^b f(x)/g(x)dx}. \quad (2.1)$$

Applying Theorem 2.1 with $w = g$ yields

$$\frac{\int_a^b g(x)/f(x)dx}{b-a} \geq \frac{\int_a^b g^2(x)dx}{\int_a^b f(x)g(x)dx},$$

giving the following inequality:

$$\frac{(b-a)\int_a^b g^2(x)dx}{\int_a^b g(x)/f(x)dx} \leq \int_a^b f(x)g(x)dx. \tag{2.2}$$

Combining Equations (2.1) and (2.2), we get

$$\frac{(b-a)\int_a^b g^2(x)dx}{\int_a^b g(x)/f(x)dx} \leq \int_a^b f(x)g(x)dx \leq \frac{(b-a)\int_a^b f^2(x)dx}{\int_a^b f(x)/g(x)dx}.$$

This completes the proof of the theorem. □

This result is of interest because integrals involving the product of two functions are classical in mathematical analysis. To the best of our knowledge, it provides a type of integral inequalities that has not previously appeared in the literature.

3. Concluding remarks

In this note, we have extended the result obtained in [2] by incorporating a weight function, and we have illustrated an application of the resulting weighted ratio-type integral inequality in a specific setting. We believe that this work can inspire further developments in the study of integral inequalities, including the derivation of more general weighted forms and their applications in analysis, probability theory, and related fields.

Acknowledgment

The author would like to thank the reviewers for their constructive comments.

References

[1] W. Liu, Q. Ngo, V. Huy: *Several interesting integral inequalities*, J. Math. Inequal., **3**(2) (2009), 201–212.

[2] V. Makanin: *Solution by V. Makanin, Sankt Petersburg, Russia*, Gazeta Matematică, seria A, **3–4** (2014), 57–58.

[3] W.T. Sulaiman: *On several interesting integral inequalities*, AIP Conf. Proc., **1281** (2010), 2234–2240.

Department of Mathematics
University of Caen-Normandie
UFR des Sciences - Campus 2, Caen
France.
Email address: christophe.chesneau@gmail.com