J COMP SCI
APPL MATH

Journal of Computer Science and Applied Mathematics
Vol. 7, no.2, (2025), 41–46
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.7.2.2

REFINEMENT ON AN INEQUALITY INVOLVING π AND E

Christophe Chesneau

ABSTRACT. In this note, we refine the famous inequality $\pi^e < e^\pi$ by adopting an integral approach. Using the same approach, we derive another elegant inequality involving π and e. The proofs require only basic integral concepts and standard logarithmic inequalities, and are of pedagogical interest. Additionally, a double inequality for the logarithmic function is established.

1. INTRODUCTION

One of the most well-known fundamental results involving mathematical constants is

$$\pi^e < e^\pi,$$

where $\pi \approx 3.141592$ and $e = \exp(1) \approx 2.718281$. This inequality reveals interesting patterns in the behavior of numbers and their exponentials. It shows that even slight differences between constants such as π and e can result in different outcomes when raised to their respective powers. This inequality has been the subject of further discussion and development in [1, 3–10].

Key words and phrases. Mathematical inequalities, mathematical constants, integrals.
Submitted: 29.11.2025; *Accepted:* 15.12.2025; *Published:* 02.01.2026.

In particular, an integral approach is considered in [6]. The first theorem in this reference can be summarized by the following equivalence:

$$\pi^e < e^\pi \quad \Leftrightarrow \quad 0 < \int_0^{\pi/e-1} \frac{t}{1+t} dt.$$

In this note, we refine this approach by introducing a variable x such that $x \geq e$ and determining a new function $f(x) \in (0, 1]$ satisfying

$$x^e \leq f(x)e^x.$$

Using f, our aim is to bridge the gap between π^e and e^π. In addition to the integral framework, a key to the proof is a standard logarithmic inequality.

We supplement this result with another elegant and new inequality. For any $x \geq e$, this inequality involves the expression $(xe)^{x+e}$, as follows:

$$e^{4x} \leq (xe)^{x+e}.$$

The proof is also based on an integral approach and a standard logarithmic inequality. As an additional contribution, it is possible to derive a new double inequality for the logarithmic function by combining some parts of the proofs.

The note is organized as follows: The results are stated and proved in Section 2. The conclusion is given in Section 3.

2. RESULTS

Our first theorem is presented below. We emphasize the variable x and the function $f(x)$.

Theorem 2.1. *For any $x \geq e$, we have*

$$x^e \leq f(x)e^x,$$

where

$$f(x) = e^{-e(\ln(x)-1)^2/2} \in (0, 1].$$

Proof. It is well known that, for any $t \geq 0$, we have $\ln(1 + t) \leq t$. Consequently, for any $x \geq e$, since the integrand is non-negative, we have

$$0 \leq \int_0^{x/e-1} \frac{t - \ln(1 + t)}{1 + t} dt.$$

Let us now evaluate this integral. Using classical primitive techniques, we obtain

$$\int_0^{x/e-1} \frac{t-\ln(1+t)}{1+t}dt = \int_0^{x/e-1} \left(1 - \frac{1}{1+t} - \frac{\ln(1+t)}{1+t}\right) dt$$
$$= \left[t - \ln(1+t) - \frac{1}{2}\left(\ln(1+t)\right)^2\right]_0^{x/e-1}$$
$$= \frac{x}{e} - 1 - \ln\left(\frac{x}{e}\right) - \frac{1}{2}\left(\ln\left(\frac{x}{e}\right)\right)^2$$
$$= \frac{x}{e} - \ln(x) - \frac{1}{2}\left(\ln(x) - 1\right)^2.$$

Using standard manipulations and the definition of f, the following equivalences hold:

$$0 \le \frac{x}{e} - \ln(x) - \frac{1}{2}\left(\ln(x) - 1\right)^2$$
$$\Leftrightarrow \quad 0 \le x - e\ln(x) - \frac{e}{2}\left(\ln(x) - 1\right)^2$$
$$\Leftrightarrow \quad \ln(x^e) \le x - \frac{e}{2}\left(\ln(x) - 1\right)^2$$
$$\Leftrightarrow \quad x^e \le e^x e^{-e(\ln(x)-1)^2/2} = f(x)e^x.$$

In addition, we have $-e\left(\ln(x) - 1\right)^2/2 \le 0$, implying that $f(x) = e^{-e(\ln(x)-1)^2/2} \in (0, 1]$, with $f(x) = 1$ only for $x = e$. This completes the proof of the theorem. □

Taking $x = \pi > e$, Theorem 2.1 gives

$$\pi^e < f(\pi)e^\pi < e^\pi.$$

Therefore, thanks to the presence of $f(\pi)$, we refine the existing inequality.

Let us perform a numerical study to support this claim. We have

$$\pi^e \approx 22.45915, \quad e^\pi \approx 23.14069, \quad f(\pi) = e^{-e(\ln(\pi)-1)^2/2} \approx 0.97193,$$

so that

$$\pi^e \approx 22.45915 < 22.49117 \approx f(\pi)e^\pi < 23.14069 \approx e^\pi.$$

Another remark concerns a part of the proof of Theorem 2.1. For any $x \ge e$, we have established that

$$0 \le x - e\ln(x) - \frac{e}{2}\left(\ln(x) - 1\right)^2,$$

which is equivalent to

$$\ln(x) \leq \sqrt{\frac{2x}{e} - 1}. \tag{2.1}$$

As far as the author knows, this is a new logarithmic inequality. It is particularly sharp for x in the neighbourhood of e.

Our second theorem is presented below. We emphasize the variable x and the function $(xe)^{x+e}$.

Theorem 2.2. *For any $x \geq e$, we have*

$$e^{4x} \leq (xe)^{x+e}.$$

Proof. It is well known that, for any $t \geq 0$, we have $\ln(1+t) \geq t/(1+t)$. Consequently, for any $x \geq e$, since the integrand is non-positive, we have

$$\int_0^{x/e-1} \left(\frac{t}{1+t} - \ln(1+t) \right) dt \leq 0.$$

Let us now evaluate this integral. Using classical primitive techniques, we obtain

$$\int_0^{x/e-1} \left(\frac{t}{1+t} - \ln(1+t) \right) dt = \int_0^{x/e-1} \left(1 - \frac{1}{1+t} - \ln(1+t) \right) dt$$
$$= [t - \ln(1+t) - ((1+t)\ln(1+t) - t)]_0^{x/e-1}$$
$$= [2t - (2+t)\ln(1+t)]_0^{x/e-1}$$
$$= 2\left(\frac{x}{e} - 1\right) - \left(\frac{x}{e} + 1\right) \ln\left(\frac{x}{e}\right)$$
$$= \frac{3x}{e} - 1 - \left(\frac{x}{e} + 1\right) \ln(x).$$

The following equivalences hold:

$$\begin{aligned} & \frac{3x}{e} - 1 - \left(\frac{x}{e} + 1\right) \ln(x) \leq 0 \\ \Leftrightarrow \quad & 3x - e - (x+e)\ln(x) \leq 0 \\ \Leftrightarrow \quad & 3x - e \leq \ln\left(x^{x+e}\right) \\ \Leftrightarrow \quad & e^{3x-e} \leq x^{x+e} \\ \Leftrightarrow \quad & e^{4x} \leq (xe)^{x+e}. \end{aligned}$$

This completes the proof of the theorem. □

In particular, taking $x = \pi > e$, Theorem 2.2 gives

$$e^{4\pi} < (\pi e)^{\pi+e}.$$

From a numerical point of view, we have

$$e^{4\pi} \approx 286751.31313 < 287175.24810 \approx (\pi e)^{\pi+e}.$$

As far as the author knows, this is a new inequality involving e and π.

Another remark concerns a part of the proof of Theorem 2.2. For any $x \geq e$, we have established that

$$\frac{3x}{e} - 1 - \left(\frac{x}{e} + 1\right)\ln(x) \leq 0,$$

which is equivalent to

$$\ln(x) \geq \frac{3x - e}{x + e}.$$

As far as the author knows, this is a new logarithmic inequality. It is particularly sharp for x in the neighbourhood of e. Combined with Equation (2.1), for any $x \geq e$, we arrive at the following elegant double inequality:

$$\frac{3x - e}{x + e} \leq \ln(x) \leq \sqrt{\frac{2x}{e} - 1}.$$

For more information on logarithmic inequalities, we refer to [2] and the references cited therein.

3. Conclusion

In this note, we contribute to the well-known inequality $\pi^e < e^\pi$ by proposing a more general and refined form. This was achieved using an integral approach and standard logarithmic inequalities. Using a similar method, we derived another elegant inequality. Additionally, we established a double inequality for the logarithmic function. We hope that these results and their comprehensive proofs will inspire further exploration of this fascinating mathematical subject.

Acknowledgment

The author would like to thank the reviewers for their constructive comments.

REFERENCES

[1] B. CHAKRABORTY: *A visual proof that* $\pi^e < e^\pi$, Math. Intell., **41**(1) (2019), 56.

[2] C. CHESNEAU: *Exploring nine different proofs of a famous logarithmic inequality*, Ann. Math. Comp. Sci., **23** (2024), 16-28.

[3] R. FARHADIAN: *A generalized form of a visual proof of* $\pi^e < e^\pi$, Math. Intell., **44**(3) (2022), 191.

[4] C. D. GALLANT: *Proof without words: Comparing* B^A *and* A^B *for* $A < B$, Math. Mag., **64**(1) (1991), 31.

[5] N. HAQUE: *A visual proof that* $e < A \Rightarrow e^A > A^e$, Math. Intell., **42**(3) (2020), 74.

[6] S. MANDAL, S. LAHIRY: *Visual proofs of the inequality* $\pi^e < e^\pi$, J. Educ. Stud. Math. Comput. Sci., **2**(1) (2025), 15–17.

[7] A. MUKHERJEE, B. CHAKRABORTY: *Yet another visual proof that* $\pi^e < e^\pi$, Math. Intell., **41**(2) (2019), 60.

[8] F. NAKHLI: *Proof without words:* $\pi^e < e^\pi$, Math. Mag., **60**(3) (1987), 165.

[9] K. NAM: *The beautiful inequality* $\pi^e < e^\pi$, Parabola, **59**(2) (2023).

[10] R. B. NELSEN: *Proof without words: Steiner's problem on the number* e, Math. Mag., **82**(2) (2009), 102.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com