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ON A COSINE-WEIGHTED VARIATION OF THE WIRTINGER INTEGRAL INEQUALITY

Christophe Chesneau

ABSTRACT. This note presents a one-parameter cosine-weighted variation of the Wirtinger integral inequality, providing a detailed proof. Additionally, two further integral inequalities are derived using a series approach.

1. INTRODUCTION

The Wirtinger integral inequality is a fundamental result in mathematical analysis. It provides a precise relationship between a function and its derivative when certain boundary conditions are met. Its simplest form is presented below. Let $f : [0, \pi] \to \mathbb{R}$ be a function such that $f \in C^1([0, \pi])$, where

$$C^1([0, \pi]) = \{f : [0, \pi] \to \mathbb{R} \mid f \text{ and } f' \text{ are continuous on } [0, \pi]\}$$

and that the following boundary conditions hold: $f(0) = f(\pi) = 0$. Then we have

$$\int_0^\pi [f(x)]^2 dx \leq \int_0^\pi [f'(x)]^2 dx.$$

This inequality plays a key role in the theory of Sobolev spaces, variational methods, and eigenvalue problems. It has also inspired numerous extensions and

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generalizations in modern mathematical analysis. See, e.g., [1–10], and the references therein.

In this note, we present a new variation of the Wirtinger integral inequality. It features a one-parameter cosine-weighted term under suitable trigonometric boundary conditions. The inequality is established via an elementary proof that relies on standard differentiation, expansion formulas and integration by parts. Building on this approach, we also derive two alternative variations of the Wirtinger integral inequality using a series-based method.

The main result is presented in Section 2, followed by two secondary results in Section 3. The note concludes with a brief discussion in Section 4.

2. MAIN RESULT

The theorem below describes our main result.

Theorem 2.1. *Let* $m \in \mathbb{N}$ *and* $f : [0, \pi] \to \mathbb{R}$ *be a function such that* $f \in C^1([0, \pi])$ *and*

$$\lim_{x\to 0} \frac{[f(x)]^2}{\sin(x)} = \lim_{x\to \pi} \frac{[f(x)]^2}{\sin(x)} = 0.$$

Then we have

$$\int_0^\pi [f(x)]^2[\cos(x)]^{2m} dx \leq \frac{1}{2m+1} \int_0^\pi [f'(x)]^2[\cos(x)]^{2m} dx.$$

Proof. For the purposes of this proof, let us introduce the intermediary function

$$g(x) = \frac{f(x)}{\sin(x)},$$

so that $f(x) = g(x)\sin(x)$. Using standard differentiation and expansion rules, we get

$$\int_0^\pi [f'(x)]^2[\cos(x)]^{2m} dx = \int_0^\pi \left[[g(x)\sin(x)]'\right]^2 [\cos(x)]^{2m} dx$$

$$= \int_0^\pi [g'(x)\sin(x) + g(x)\cos(x)]^2[\cos(x)]^{2m} dx$$

$$= \int_0^\pi [g'(x)]^2[\sin(x)]^2[\cos(x)]^{2m} dx + 2\int_0^\pi g'(x)g(x)\sin(x)[\cos(x)]^{2m+1} dx$$

$$+ \int_0^\pi [g(x)]^2[\cos(x)]^{2m+2} dx. \tag{2.1}$$

Let us now focus on the second integral term, i.e., $2\int_0^\pi g'(x)g(x)\sin(x)[\cos(x)]^{2m+1}dx$. Using an integration by parts based on $2g'(x)g(x) = [[g(x)]^2]'$, the definition of g, and the boundary conditions, we have

$$\begin{aligned}
&2\int_0^\pi g'(x)g(x)\sin(x)[\cos(x)]^{2m+1}dx = \left[[g(x)]^2\sin(x)[\cos(x)]^{2m+1}\right]_{x\to 0}^{x\to\pi} \\
&- \int_0^\pi [g(x)]^2\left[[\cos(x)]^{2m+2} - (2m+1)[\sin(x)]^2[\cos(x)]^{2m}\right]dx \\
&= -\lim_{x\to\pi}\frac{[f(x)]^2}{\sin(x)} - \lim_{x\to 0}\frac{[f(x)]^2}{\sin(x)} \\
&- \int_0^\pi [g(x)]^2[\cos(x)]^{2m+2}dx + (2m+1)\int_0^\pi [g(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx \\
&= -\int_0^\pi [g(x)]^2[\cos(x)]^{2m+2}dx + (2m+1)\int_0^\pi [g(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx. \qquad (2.2)
\end{aligned}$$

Combining Equations (2.1) and (2.2), simplifying $\int_0^\pi [g(x)]^2[\cos(x)]^{2m+2}dx$ and its negative, using the fact that $\int_0^\pi [g'(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx \geq 0$ and expressing g, we obtain

$$\begin{aligned}
&\int_0^\pi [f'(x)]^2[\cos(x)]^{2m}dx \\
&= \int_0^\pi [g'(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx + (2m+1)\int_0^\pi [g(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx \\
&\geq (2m+1)\int_0^\pi [g(x)]^2[\sin(x)]^2[\cos(x)]^{2m}dx \\
&= (2m+1)\int_0^\pi [f(x)]^2[\cos(x)]^{2m}dx.
\end{aligned}$$

This implies that

$$\int_0^\pi [f(x)]^2[\cos(x)]^{2m}dx \leq \frac{1}{2m+1}\int_0^\pi [f'(x)]^2[\cos(x)]^{2m}dx.$$

The proof is completed. □

Theorem 2.1 applies to a wide class of functions. In particular, simple examples satisfying the boundary conditions

$$\lim_{x\to 0}\frac{[f(x)]^2}{\sin(x)} = \lim_{x\to\pi}\frac{[f(x)]^2}{\sin(x)} = 0,$$

include

- $f(x) = x^\alpha(\pi - x)^\beta$ with $\alpha, \beta > 1/2$,
- $f(x) = [\sin(x)]^\alpha[\sin(\pi - x)]^\beta$ with $\alpha, \beta > 1/2$,
- $f(x) = x^\alpha[\sin(\pi - x)]^\beta$ with $\alpha, \beta > 1/2$,
- $f(x) = [\sin(x)]^\alpha(\pi - x)^\beta$ with $\alpha, \beta > 1/2$,
- $f(x) = \sqrt{x(\pi - x)}e^{-1/[x(\pi-x)]}$.

Some specific inequalities that follow from Theorem 2.1 are presented below.

- Setting $m = 0$, Theorem 2.1 reduces to the classical Wirtinger integral inequality, but under the indicated trigonometric boundary conditions.
- Setting $m = 1$, we get

$$\int_0^\pi [f(x)]^2[\cos(x)]^2 dx \leq \frac{1}{3}\int_0^\pi [f'(x)]^2[\cos(x)]^2 dx.$$

- Setting $m = 2$, we obtain

$$\int_0^\pi [f(x)]^2[\cos(x)]^4 dx \leq \frac{1}{5}\int_0^\pi [f'(x)]^2[\cos(x)]^4 dx.$$

Additional examples can be formulated in the same way, all apparently new within the existing literature.

The proposition below demonstrates that the boundary conditions of the classical Wirtinger integral inequality imply those considered in Theorem 2.1.

Proposition 2.1. *Let* $f : [0, \pi] \to \mathbb{R}$ *be a function such that* $f \in C^1([0, \pi])$ *and* $f(0) = f(\pi) = 0$. *Then we have*

$$\lim_{x\to 0} \frac{[f(x)]^2}{\sin(x)} = \lim_{x\to \pi} \frac{[f(x)]^2}{\sin(x)} = 0.$$

Proof. Since $f \in C^1([0, \pi])$ and $f(0) = 0$, for any $x \in (0, \pi)$, we can write

$$f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x f'(t)dt.$$

Applying the Cauchy-Schwarz integral inequality, we get

$$|f(x)| = \left|\int_0^x f'(t)dt\right| \leq \sqrt{\int_0^x dt}\sqrt{\int_0^x [f'(t)]^2 dt} = \sqrt{x}\sqrt{\int_0^x [f'(t)]^2 dt}.$$

Therefore, for any $x \in (0, \pi)$, we have

$$0 \leq \frac{[f(x)]^2}{\sin(x)} \leq \frac{x}{\sin(x)}\int_0^x [f'(t)]^2 dt.$$

Since $\lim_{x\to 0}\sin(x)/x = 1$ and $\lim_{x\to 0}\int_0^x [f'(t)]^2 dt = 0$, we have

$$0 \le \lim_{x\to 0}\frac{[f(x)]^2}{\sin(x)} \le \lim_{x\to 0}\frac{x}{\sin(x)}\int_0^x [f'(t)]^2 dt = 1\times 0 = 0,$$

so that

$$\lim_{x\to 0}\frac{[f(x)]^2}{\sin(x)} = 0.$$

The first limit result has been obtained. Let us now examine the other one using a similar method. Since $f \in C^1([0,\pi])$ and $f(\pi) = 0$, for any $x \in (0,\pi)$, we can write

$$f(x) = f(\pi) - \int_x^\pi f'(t)dt = -\int_x^\pi f'(t)dt.$$

Applying the Cauchy-Schwarz integral inequality, we get

$$|f(x)| = \left|\int_x^\pi f'(t)dt\right| \le \sqrt{\int_x^\pi dt}\sqrt{\int_x^\pi [f'(t)]^2 dt} = \sqrt{\pi - x}\sqrt{\int_x^\pi [f'(t)]^2 dt}.$$

Therefore, for any $x \in (0,\pi)$, we have

$$0 \le \frac{[f(x)]^2}{\sin(x)} \le \frac{\pi - x}{\sin(x)}\int_x^\pi [f'(t)]^2 dt.$$

Since $\lim_{x\to\pi}\sin(x)/(\pi - x) = 1$ and $\lim_{x\to\pi}\int_x^\pi [f'(t)]^2 dt = 0$, we have

$$0 \le \lim_{x\to\pi}\frac{[f(x)]^2}{\sin(x)} \le \lim_{x\to\pi}\frac{\pi - x}{\sin(x)}\int_x^\pi [f'(t)]^2 dt = 1\times 0 = 0,$$

so that

$$\lim_{x\to\pi}\frac{[f(x)]^2}{\sin(x)} = 0.$$

This completes the proof. □

Consequently, Theorem 2.1 holds under the boundary conditions of the classical Wirtinger integral inequality. A rigorous formulation of this result is presented below.

Theorem 2.2. *Let $m \in \mathbb{N}$ and $f : [0,\pi] \to \mathbb{R}$ be a function such that $f \in C^1([0,\pi])$ and $f(0) = f(\pi) = 0$. Then we have*

$$\int_0^\pi [f(x)]^2[\cos(x)]^{2m}dx \le \frac{1}{2m+1}\int_0^\pi [f'(x)]^2[\cos(x)]^{2m}dx.$$

The rest of the note presents two secondary results derived from Theorem 2.1.

3. Secondary results

A consequence of Theorem 2.1 is the theorem below, which is based on a direct series approach.

Theorem 3.1. *Let $f : [0, \pi] \to \mathbb{R}$ be a function such that $f \in C^1([0, \pi])$ and*

$$\lim_{x\to 0} \frac{[f(x)]^2}{\sin(x)} = \lim_{x\to \pi} \frac{[f(x)]^2}{\sin(x)} = 0.$$

Then we have

$$\int_0^{\pi} [f(x)]^2 \frac{1}{[\sin(x)]^2} dx \leq \frac{1}{2} \int_0^{\pi} [f'(x)]^2 \frac{1}{\cos(x)} \log\left(\frac{1+\cos(x)}{1-\cos(x)}\right) dx,$$

provided that the integrals exist.

Proof. It follows from Theorem 2.1 that, for any $m \in \mathbb{N}$,

$$\int_0^{\pi} [f(x)]^2 [\cos(x)]^{2m} dx \leq \frac{1}{2m+1} \int_0^{\pi} [f'(x)]^2 [\cos(x)]^{2m} dx.$$

Summing both sides with respect to $m \in \mathbb{N}$, we get

$$\sum_{m=0}^{\infty} \left[\int_0^{\pi} [f(x)]^2 [\cos(x)]^{2m} dx\right] \leq \sum_{m=0}^{\infty} \left[\frac{1}{2m+1} \int_0^{\pi} [f'(x)]^2 [\cos(x)]^{2m} dx\right].$$

By the Lebesgue dominated convergence theorem, we can interchange the sum and the integral, yielding

$$\int_0^{\pi} [f(x)]^2 \sum_{m=0}^{\infty} [\cos(x)]^{2m} dx \leq \int_0^{\pi} [f'(x)]^2 \sum_{m=0}^{\infty} \frac{1}{2m+1} [\cos(x)]^{2m} dx.$$

Using the classical geometric series for the left term taking into account that $[\cos(x)]^2 \in (0, 1)$ for any $x \in (0, \pi)$ almost surely, and the difference of logarithmic series for the right term, i.e., for any $y \in (-1, 1)$,

$$\sum_{m=0}^{\infty} \frac{y^{2m+1}}{2m+1} = \frac{1}{2} \log\left(\frac{1+y}{1-y}\right),$$

so that, for $y \neq 0$,

$$\sum_{m=0}^{\infty} \frac{y^{2m}}{2m+1} = \frac{1}{2y} \log\left(\frac{1+y}{1-y}\right),$$

(also with reference to the hyperbolic arctangent function), taking into account that $\cos(x) \in (-1,1)$ for any $x \in (0,\pi)$ almost surely, we get

$$\int_0^{\pi} [f(x)]^2 \frac{1}{1-[\cos(x)]^2} dx \leq \int_0^{\pi} [f'(x)]^2 \frac{1}{2\cos(x)} \log\left(\frac{1+\cos(x)}{1-\cos(x)}\right) dx,$$

so that

$$\int_0^{\pi} [f(x)]^2 \frac{1}{[\sin(x)]^2} dx \leq \frac{1}{2} \int_0^{\pi} [f'(x)]^2 \frac{1}{\cos(x)} \log\left(\frac{1+\cos(x)}{1-\cos(x)}\right) dx.$$

This completes the proof. □

For the special case $f(x) = \sin(x)$, one can prove that both sides of the inequalities are equal to π.

Another consequence of Theorem 2.1 is the theorem below, which is still based on a direct series approach.

Theorem 3.2. *Let $f : [0,\pi] \to \mathbb{R}$ be a function such that $f \in C^1([0,\pi])$ and*

$$\lim_{x\to 0} \frac{[f(x)]^2}{\sin(x)} = \lim_{x\to\pi} \frac{[f(x)]^2}{\sin(x)} = 0.$$

Then we have

$$\int_0^{\pi} [f(x)]^2 \frac{1+[\cos(x)]^2}{[\sin(x)]^4} dx \leq \int_0^{\pi} [f'(x)]^2 \frac{1}{[\sin(x)]^2} dx,$$

provided that the integrals exist.

Proof. It follows from Theorem 2.1 that, for any $m \in \mathbb{N}$,

$$\int_0^{\pi} [f(x)]^2 [\cos(x)]^{2m} dx \leq \frac{1}{2m+1} \int_0^{\pi} [f'(x)]^2 [\cos(x)]^{2m} dx,$$

so that

$$(2m+1) \int_0^{\pi} [f(x)]^2 [\cos(x)]^{2m} dx \leq \int_0^{\pi} [f'(x)]^2 [\cos(x)]^{2m} dx.$$

Summing both sides with respect to $m \in \mathbb{N}$, we get

$$\sum_{m=0}^{\infty} \left[(2m+1) \int_0^{\pi} [f(x)]^2 [\cos(x)]^{2m} dx\right] \leq \sum_{m=0}^{\infty} \left[\int_0^{\pi} [f'(x)]^2 [\cos(x)]^{2m} dx\right].$$

By the Lebesgue dominated convergence theorem, we can interchange the sum and the integral, yielding

$$\int_0^\pi [f(x)]^2 \sum_{m=0}^{\infty}(2m+1)[\cos(x)]^{2m}dx \leq \int_0^\pi [f'(x)]^2 \sum_{m=0}^{\infty}[\cos(x)]^{2m}dx.$$

Using the geometric-type series, for any $y \in (-1, 1)$,

$$\sum_{m=0}^{\infty}(2m+1)y^{2m} = \frac{1+y^2}{(1-y^2)^2}$$

for the left term and the classical geometric series for the right term, taking into account that $[\cos(x)]^2 \in (0, 1)$ for any $x \in (0, \pi)$ almost surely, we get

$$\int_0^\pi [f(x)]^2 \frac{1+[\cos(x)]^2}{[1-[\cos(x)]^2]^2}dx \leq \int_0^\pi [f'(x)]^2 \frac{1}{1-[\cos(x)]^2}dx,$$

so that

$$\int_0^\pi [f(x)]^2 \frac{1+[\cos(x)]^2}{[\sin(x)]^4}dx \leq \int_0^\pi [f'(x)]^2 \frac{1}{[\sin(x)]^2}dx.$$

This completes the proof. □

As far as the author is aware, the variations of the Wirtinger integral inequality in Theorems 3.1 and 3.2 are original.

4. CONCLUSION

In this note, we introduced a one-parameter cosine-weighted variation of the Wirtinger integral inequality, with a proof based on standard integral techniques. The result accommodates a broad class of functions. Two additional variations are derived via a series-based approach. Future research may focus on multidimensional extensions, the determination of sharper constants, and applications to weighted inequalities in differential equations and functional analysis.

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Conflict of interest

The author declares that there is no conflict of interest related to this paper.

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Department of Mathematics
University of Caen-Normandie
UFR des Sciences - Campus 2, Caen
France.
Email address: christophe.chesneau@gmail.com