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ON TWO CUMULATIVE RIEMANN SUM INTEGRAL INEQUALITIES IN THE CONTEXT OF CONVEX AND CONCAVE FUNCTIONS

Christophe Chesneau

ABSTRACT. Motivated by recent developments in cumulative Riemann sum integral inequalities, this article addresses the problem in the setting of convex and concave functions. We establish two new theorems. The proofs are presented in detail to ensure both clarity and rigor. Furthermore, several examples are provided to illustrate the obtained theorems.

1. INTRODUCTION

Convex and concave functions are classical classes of functions in mathematics, as recalled below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ such that $b > a$. Then $f : [a, b] \to \mathbb{R}$ is a convex function if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Similarly, $f : [a, b] \to \mathbb{R}$ is a concave function if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).$$

One of the most famous integral inequalities related to these classes of functions is the Hermite-Hadamard integral inequality, as recalled below. Let $a, b \in \mathbb{R}$ such

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that $b > a$. Then, assuming that $f : [a, b] \to \mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Similarly, assuming that $f : [a, b] \to \mathbb{R}$ is a concave function, we have

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right). \tag{1.2}$$

The Hermite-Hadamard integral inequality has inspired numerous extensions and generalizations. We refer to [1–15, 17].

Inspired by [16], which addresses decreasing functions, this article aims to establish two cumulative Riemann sum integral inequalities in the setting of convex and concave functions. To achieve this, we show that the Hermite–Hadamard integral inequality can be used in an efficient and natural way. Several examples involving convex and concave functions are provided to illustrate the results.

The remainder of the article is organized as follows: The next section presents the main theorems. Section 3 concludes the article and discusses possible directions for future research.

2. MAIN THEOREMS

2.1. **A theorem in [16].** The key theorem in [16] is recalled below.

Theorem 2.1. *[16, Theorem 2.1] Let* $n \in \mathbb{N}\backslash\{0\}$, $\theta_1, \ldots, \theta_n > 0$ *with*

$$\sum_{i=1}^{n} \theta_i = 1$$

and, for any $i = 1, \ldots, n$,

$$S_i = \sum_{j=1}^{i} \theta_j.$$

We also set $S_0 = 0$. *Let* $g : [0, 1] \to \mathbb{R}$ *be a decreasing integrable function. Then we have*

$$\sum_{i=1}^{n} \theta_i g(S_i) \leq \int_0^1 g(x)dx.$$

The proof consists in expressing θ_i in terms of S_i and S_{i-1}, and then exploiting the monotonicity of g, together with suitable summation arguments. Several illustrative examples are provided in [16], including exponential, logarithmic, reciprocal, and trigonometric decreasing functions.

In this article, we extend this result to the broader framework of convex and concave functions. More precisely, we establish two new theorems, which are presented in the subsection below.

2.2. **New theorems.** A cumulative Riemann sum integral inequality in the context of convex functions is given in the theorem below.

Theorem 2.2. *Let* $n \in \mathbb{N}\backslash\{0\}$*,* $\theta_1, \ldots, \theta_n > 0$ *with*

$$\sum_{i=1}^{n} \theta_i = 1$$

and, for any $i = 1, \ldots, n$*,*

$$S_i = \sum_{j=1}^{i} \theta_j.$$

We also set $S_0 = 0$*. Let* $g : [0, 1] \to \mathbb{R}$ *be a convex integrable function. Then we have*

$$\sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) \leq \int_0^1 g(x)dx.$$

Proof. For any $i = 1, \ldots, n$, we have

$$\theta_i = S_i - S_{i-1}$$

and

$$S_i - \frac{\theta_i}{2} = S_{i-1} + \frac{\theta_i}{2} = \frac{2S_{i-1} + \theta_i}{2} = \frac{S_{i-1} + (S_{i-1} + \theta_i)}{2} = \frac{S_{i-1} + S_i}{2}.$$

Therefore, we can write

$$\sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) = \sum_{i=1}^{n} (S_i - S_{i-1})\, g\left(\frac{S_{i-1} + S_i}{2}\right).$$

Applying the left-hand side of the Hermite-Hadamard integral inequality in Equation (1.1) to the convex function g with $a = S_{i-1}$ and $b = S_i$, we get

$$(S_i - S_{i-1})\, g\left(\frac{S_{i-1} + S_i}{2}\right) \leq \int_{S_{i-1}}^{S_i} g(x)dx.$$

Summing with respect to i with $i = 1, \ldots, n$ and using the Chasles integral relation, $S_0 = 0$ and $S_n = 1$, yields

$$\sum_{i=1}^{n} (S_i - S_{i-1})\, g\left(\frac{S_{i-1} + S_i}{2}\right) \leq \sum_{i=1}^{n} \int_{S_{i-1}}^{S_i} g(x)dx = \int_{S_0}^{S_n} g(x)dx = \int_0^1 g(x)dx.$$

Therefore, combining the obtained inequalities, we get

$$\sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) \leq \int_0^1 g(x)dx.$$

This completes the proof of the theorem. □

Some examples of applications of Theorem 2.2 are given below.

- If we take $g(x) = x^2$, which is convex, we have

$$\sum_{i=1}^{n} \theta_i \left(S_i - \frac{\theta_i}{2}\right)^2 = \sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) \leq \int_0^1 g(x)dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

- If we take $g(x) = e^x$, which is convex, we have

$$\sum_{i=1}^{n} \theta_i e^{S_i - \frac{\theta_i}{2}} = \sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) \leq \int_0^1 g(x)dx = \int_0^1 e^x dx = e - 1.$$

- If we take $g(x) = \log(1 + e^x)$, which is convex, we have

$$\sum_{i=1}^{n} \theta_i \log\left(1 + e^{S_i - \frac{\theta_i}{2}}\right) = \sum_{i=1}^{n} \theta_i g\left(S_i - \frac{\theta_i}{2}\right) \leq \int_0^1 g(x)dx$$
$$= \int_0^1 \log(1 + e^x)dx \approx 0.983819.$$

The theorem below can be presented as the concave analogue to Theorem 2.2.

Theorem 2.3. *Let* $n \in \mathbb{N}\backslash\{0\}$, $\theta_1, \ldots, \theta_n > 0$ *with*

$$\sum_{i=1}^{n} \theta_i = 1$$

and, for any $i = 1, \ldots, n$,

$$S_i = \sum_{j=1}^{i} \theta_j.$$

We also set $S_0 = 0$. *Let* $g : [0, 1] \to \mathbb{R}$ *be a concave integrable function. Then we have*

$$\sum_{i=1}^{n} \theta_i g(S_i) + \sum_{i=1}^{n} \theta_i g(S_{i-1}) \le 2 \int_0^1 g(x)dx.$$

Proof. For any $i = 1, \ldots, n$, we have

$$\theta_i = S_i - S_{i-1}.$$

Moreover, the inequality

$$\sum_{i=1}^{n} \theta_i g(S_i) + \sum_{i=1}^{n} \theta_i g(S_{i-1}) \le 2 \int_0^1 g(x)dx$$

is equivalent to

$$\sum_{i=1}^{n} \theta_i \frac{g(S_i) + g(S_{i-1})}{2} \le \int_0^1 g(x)dx.$$

Therefore, we can write

$$\sum_{i=1}^{n} \theta_i \frac{g(S_i) + g(S_{i-1})}{2} = \sum_{i=1}^{n} (S_i - S_{i-1}) \frac{g(S_i) + g(S_{i-1})}{2}.$$

Applying the left-hand side of the Hermite-Hadamard integral inequality in Equation (1.2) to the concave function g with $a = S_{i-1}$ and $b = S_i$, we get

$$(S_i - S_{i-1}) \frac{g(S_i) + g(S_{i-1})}{2} \le \int_{S_{i-1}}^{S_i} g(x)dx.$$

Summing with respect to i with $i = 1, \ldots, n$ and using the Chasles integral relation, $S_0 = 0$ and $S_n = 1$, yields

$$\sum_{i=1}^{n} (S_i - S_{i-1}) \frac{g(S_i) + g(S_{i-1})}{2} \le \sum_{i=1}^{n} \int_{S_{i-1}}^{S_i} g(x)dx = \int_{S_0}^{S_n} g(x)dx = \int_0^1 g(x)dx.$$

Therefore, combining the obtained inequalities, we get

$$\sum_{i=1}^{n} \theta_i g(S_i) + \sum_{i=1}^{n} \theta_i g(S_{i-1}) \le 2 \int_0^1 g(x)dx.$$

This completes the proof of the theorem. □

Some examples of applications of Theorem 2.3 are given below.

- If we take $g(x) = \sqrt{x}$, which is concave, we have

$$\sum_{i=1}^{n} \theta_i \sqrt{S_i} + \sum_{i=1}^{n} \theta_i \sqrt{S_{i-1}} = \sum_{i=1}^{n} \theta_i g\left(S_i\right) + \sum_{i=1}^{n} \theta_i g\left(S_{i-1}\right)$$
$$\leq 2 \int_0^1 g(x)dx = 2 \int_0^1 \sqrt{x}dx = \frac{4}{3}.$$

- If we take $g(x) = \log(1+x)$, which is concave, we have

$$\sum_{i=1}^{n} \theta_i \log\left(1+S_i\right) + \sum_{i=1}^{n} \theta_i \log\left(1+S_{i-1}\right) = \sum_{i=1}^{n} \theta_i g\left(S_i\right) + \sum_{i=1}^{n} \theta_i g\left(S_{i-1}\right)$$
$$\leq 2 \int_0^1 g(x)dx = 2 \int_0^1 \log(1+x)dx = 2\left(\log(4) - 1\right).$$

This implies that

$$\log\left(\prod_{i=1}^{n} \left(1+S_i\right)^{\theta_i} \left(1+S_{i-1}\right)^{\theta_i}\right) \leq 2\left(\log(4) - 1\right),$$

which is equivalent to

$$\prod_{i=1}^{n} \left(1+S_i\right)^{\theta_i} \left(1+S_{i-1}\right)^{\theta_i} \leq 16e^{-2}.$$

- If we take $g(x) = \arctan(x)$, which is concave, we have

$$\sum_{i=1}^{n} \theta_i \arctan\left(S_i\right) + \sum_{i=1}^{n} \theta_i \arctan\left(S_{i-1}\right) = \sum_{i=1}^{n} \theta_i g\left(S_i\right) + \sum_{i=1}^{n} \theta_i g\left(S_{i-1}\right)$$
$$\leq 2 \int_0^1 g(x)dx = 2 \int_0^1 \arctan(x)dx = \frac{1}{2}\left(\pi - \log(4)\right).$$

Note that, in all our examples, we consider increasing functions g, which departs from the setting of [16].

3. CONCLUSION

In this article, inspired by [16], we establish two integral inequalities involving cumulative Riemann sums in the setting of convex and concave functions. The proofs are based on the Hermite-Hadamard integral inequality, which serves as the central analytical tool. Detailed arguments are provided to ensure clarity and rigor. Several illustrative examples are also included to demonstrate the applicability of the obtained results.

Some perspectives for future research include the extension of these inequalities to broader classes of functions, such as quasi-convex or higher-order convex functions, as well as their adaptation to multidimensional settings. Another promising direction is the investigation of analogous inequalities within discrete frameworks or under alternative notions of integration. Finally, potential applications in numerical analysis and optimization theory could be explored.

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REFERENCES

[1] E.F. BECKENBACH: *Convex functions*, Bull. Amer. Math. Soc., **54** (1948), 439–460.

[2] R. BELLMAN: *On the approximation of curves by line segments using dynamic programming*, Commun. ACM, **4**(6) (1961), 284.

[3] C. CHESNEAU: *On several new integral convex theorems*, Adv. Math. Sci. J., **14**(4) (2025), 391–404.

[4] C. CHESNEAU: *Examining new convex integral inequalities*, Earthline J. Math. Sci., **15**(6) (2025), 1043–1049.

[5] C. CHESNEAU: *On two new theorems on convex integral inequalities*, Adv. Math. Sci. J., **15**(1) (2026), 67–75.

[6] S.S.DRAGOMIR, R.P. AGARWAL: *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math.Lett. **11**(5) (1998), 91-95.

[7] J. HADAMARD: *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

[8] C. HERMITE: *Sur deux limites d'une intégrale définie*, Mathesis, **3** (1883), 82.

[9] M.M. IDDRISU, C.A. OKPOTI, K.A. GBOLAGADE: *A proof of Jensen's inequality through a new Steffensen's inequality*, Adv. Inequal. Appl., **2014** (2014), 1–7.

[10] M.M. IDDRISU, C.A. OKPOTI, K.A. GBOLAGADE: *Geometrical proof of new Steffensen's inequality and Applications*, Adv. Inequal. Appl., **2014** (2014), 1–10.

[11] J.L.W.V. JENSEN: *Om konvekse Funktioner og Uligheder mellem Middelvaerdier*, Nyt Tidsskr. Math. B., **16** (1905), 49–68.

[12] J.L.W.V. JENSEN: *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math., **30** (1906), 175–193.

[13] D.S. MITRINOVIĆ: *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[14] D.S. MITRINOVIĆ, J.E. PEČARIĆ, A.M. FINK: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[15] C.P. NICULESCU: *Convexity according to the geometric mean*, Math. Ineq. Appl., **3**(2) (2000), 155–167.

[16] J.-C. PAIN: Cumulative Riemann sums, distribution functions, and a universal inequality, Arxiv, arXiv:2603.08959, (2026), 1-8.

[17] A.W. ROBERTS, P.E. VARBERG: *Convex Functions*, Academic Press, 1973.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: `christophe.chesneau@gmail.com`