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A NOTE ON A TWO-SIDED INTEGRAL INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. This article is devoted to establishing a two-sided integral inequality for convex functions that generalizes the Hermite-Hadamard integral inequality. The generalization is achieved through the introduction of an auxiliary function. A detailed proof is provided.

1. INTRODUCTION

Convex functions are fundamental in mathematics and its applications. For completeness, we recall the definition of a convex function below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for every $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

As a notable property, if f is twice differentiable and convex, then we have $f''(x) \geq 0$ for every $x \in [a, b]$, implying that f' is increasing. A key consequence of convexity is the emergence of a rich family of integral inequalities, often referred to as convex integral inequalities. Perhaps the most notable and classical example is the Hermite-Hadamard integral inequality, which we state below. Let

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$a, b \in \mathbb{R}$ such that $b > a$. Assuming that $f : [a, b] \to \mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

In recent decades, considerable advances have been made, as numerous authors have explored and established profound links between convexity, functional inequalities, and integral transforms. Detailed expositions can be found in [1–17].

The purpose of this article is to derive a two-sided integral inequality for convex functions, extending the classical Hermite-Hadamard integral inequality. The proposed generalization is obtained by incorporating an auxiliary weight function, denoted w. The main result is established through a complete and rigorous proof.

The remainder of the article is composed of three sections: Section 2 describes the main result, followed by the proof and some remarks. Section 3 contains the conclusion.

2. Result

Our new two-sided integral inequality for convex functions is stated in the theorem below.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \to \mathbb{R}$ be a convex function, and $w : [a, b] \to [0, \infty)$ be an integrable function such that*

$$\int_a^b w(x)dx = 1.$$

Let us define

$$\mu = \int_a^b xw(x)dx,$$

provided that it exists. Then the following two-sided inequality holds:

$$f(\mu) \leq \int_a^b f(x)w(x)dx \leq \frac{b-\mu}{b-a}f(a) + \frac{\mu-a}{b-a}f(b).$$

Proof. Since f is convex and w is a probability density function on $[a, b]$, the Jensen integral inequality directly yields

$$f\left(\int_a^b xw(x)dx\right) \leq \int_a^b f(x)w(x)dx.$$

By the definition of μ, this becomes

$$f(\mu) \leq \int_a^b f(x)w(x)dx.$$

The lower bound is established.

Let us now prove the upper bound. For every $x \in [a,b]$, we can write

$$x = \frac{(b-a)x}{b-a} = \frac{(b-a)x - ab + ab}{b-a} = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b,$$

with $(b-x)/(b-a) \in [0,1]$, $(x-a)/(b-a) \in [0,1]$ and

$$\frac{b-x}{b-a} + \frac{x-a}{b-a} = \frac{b-a}{b-a} = 1.$$

By the convexity of f, for every $x \in [a,b]$, we have

$$f(x) = f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b).$$

Multiplying both sides by $w(x)$ and integrating over $[a,b]$, we obtain

$$\int_a^b f(x)w(x)dx \leq \int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\right) w(x)dx.$$

We have

$$\int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\right) w(x)dx$$
$$= f(a)\frac{1}{b-a}\int_a^b (b-x)w(x)dx + f(b)\frac{1}{b-a}\int_a^b (x-a)w(x)dx.$$

Using the definition of μ, using $\int_a^b w(x)dx = 1$, we compute

$$\int_a^b (b-x)w(x)dx = b\int_a^b w(x)dx - \int_a^b xw(x)dx = b - \mu$$

and

$$\int_a^b (x-a)w(x)dx = \int_a^b xw(x)dx - a\int_a^b w(x)dx = \mu - a.$$

Combining the above inequalities, we get

$$\int_a^b f(x)w(x)dx \le \frac{b-\mu}{b-a}f(a) + \frac{\mu-a}{b-a}f(b).$$

Combining both bounds completes the proof. □

Some remarks on this theorem are given below.

Remark 2.1. *In Theorem 2.1, if*

$$w(x) = \frac{1}{b-a}$$

is the uniform weight, then

$$\mu = \frac{1}{b-a}\int_a^b x dx = \frac{a+b}{2}$$

and the two-sided inequality reduces to

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

which is the classical Hermite-Hadamard integral inequality.

Remark 2.2. *Theorem 2.1 may have the following probabilistic interpretation. Let $f : [a,b] \to \mathbb{R}$ be a convex function, and X be a random variable with probability density function $w : [a,b] \to [0,\infty)$. Let $\mathbb{E}$ be the expectation operator. Then the following two-sided inequality holds:*

$$f(\mathbb{E}(X)) \le \mathbb{E}(f(X)) \le \frac{b-\mathbb{E}(X)}{b-a}f(a) + \frac{\mathbb{E}(X)-a}{b-a}f(b),$$

provided that $\mathbb{E}(X)$ exists.

Remark 2.3. *In Theorem 2.1, if f is concave rather than convex, then the two-sided inequality is reversed.*

3. Conclusion

In this article, we established a new two-sided integral inequality for convex functions that extends the classical Hermite-Hadamard integral inequality through the introduction of an auxiliary weight function. The result provides a flexible framework for deriving refined bounds and highlights the role of weighted averages in the study of convex integral inequalities. Future work may explore extensions to higher-dimensional settings, applications to related classes of functions such as concave or quasi-convex functions, and further connections with functional inequalities and integral transforms.

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